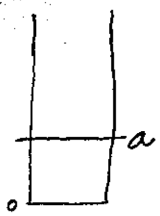
Theorem Generalized to any level



$$\nu \overline{U}_y = \overline{VU}$$

$$\therefore \nu \overline{U}(a) = -\int_a^\infty \overline{VU}\,dy = -\int_a^\infty \oint dx\,dy\; V(x,y)\,U(x,y)\,dy$$

$$V(x,\infty) = \int_0^\infty V_y\,dy = A(x)$$

But $V(x,y) = -\int_y^\infty V_y(x,y')\,dy' + V(x,\infty)$

$$= +\int_y^\infty U_x(x,y')\,dy' + A(x)$$

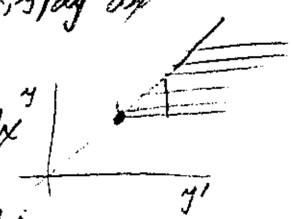
$$\therefore \nu \overline{U}(a) = -\oint\int_a^\infty dy \int_y^\infty dy'\, U_x(x,y')\,dx\;\, U(x,y) \;-\; \underbrace{\int_a^\infty \oint dx\,dy\,A(x)\,U(x,y)}_{I_1}$$

$$\nu \overline{U}(a) + I_1 = -\oint\int_a^\infty dy\; U(x,y)\int_y^\infty \left[\nu U_{yy} - \tfrac{\partial}{\partial x}(U^2) - \tfrac{\partial}{\partial y}(VU)\right] dy'\,dx$$

Reconsider $\int dy' \int_a^{y'} dy$

$$= +\int_a^\infty \oint dy\left[\nu U_y U - VU^2\right] + \int_a^\infty dy\; V_y(x,y)\int_y^\infty U^2(x,y')\,dy'\,dx$$

$$= -\tfrac{1}{2}\nu\,\overline{U^2(a)} - \int_a^\infty \oint dy\; V\overline{U}^2 + \int_a U^2(x,y')\,V(x,y')\,dy'\,dx$$



$$-\int_a^\infty U^2(x,y')\,V(x,a)\,dy'\,dx$$

$$\therefore \nu\left[\overline{U(a)} + \tfrac{1}{2}\overline{U^2(a)}\right] = \oint\left(\int_a^\infty U^2(x,y')\,dy'\right)V(x,a)\,dx - \int_a^\infty \oint dx\; U(x,y)\,A(x)\,dy$$

Now if $\int_a^\infty U(x,y)\,dy = B(x)$, then $B_x(x) = \int_a^\infty U_x(x,y)\,dy = -\int_a^\infty V_y(x,y)\,dy = -A(x) + V(x,a)$.

$$\therefore A(x) = V(x,a) - B_x(x)$$

But the last integral is $\oint B(x)A(x)\,dx = \int B(x)\,V(x,a)\,dx$, since $\overline{B(x)B_x(x)} = 0$

$$\therefore \nu\left[U(a) + \tfrac{1}{2}U^2(a)\right] = -\oint dx\; V(x,a)\int_a^\infty \left[U^2(x,y') + U(x,y')\right] dy'$$

$u = U + u \qquad \nu\omega_y = v\omega - \Pi_x \qquad (1)$

$v = v \qquad \nu\omega_x = U\omega + u\omega + \Pi_y \qquad (2)$

take as zero.

$$\Pi = \overbrace{p + \tfrac{1}{2}U^2}^{} + Uu + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2$$

$$= Uu + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2.$$

av (1) $\quad \nu\overline{\omega_y} = \overline{v\omega}$ or $\nu\overline{u_{yy}} = \overline{vu_y}$ or $\overline{\nu u_y} = \overline{vu}$

$$\therefore \nu f = -\int_0^\infty V(x,y)U(x,y)\,dx\,dy = -\int_0^\infty u(x,y)\int_y^\infty u_x(x,y')\,dy'\,dy\,dx$$

$\cancel{\delta p = p + \tfrac{1}{2}U^2}$

$\cancel{-(p+\tfrac{1}{2}U^2)_\infty}$

Consider $U\int_y^\infty u_x(x,y')\,dy' = \int_y^\infty \Pi_x\,dy' - \int_y^\infty (uu_x + vv_x)\,dy' - \int_y^\infty (\delta p)_x\,dy$

$$= +\nu\omega + \int_y^\infty v(v_x - u_y)\,dy' + \int (uv_y - \cancel{vv_x})\,dy' - \int_y^\infty p_x\,dy$$

$$= \nu\omega + \int_y^\infty (vu_y - uv_y)\,dy' - \int_y^\infty p_x\,dy$$

uv_x

$$= \nu(v_x - u_y) - vu + \int_y^\infty \frac{\partial}{\partial x}(u^2(x,y'))\,dy' - \int_y^\infty p_x\,dy$$

$p_\infty = 0$

$$\therefore \nu\overline{f} = \int_0^\infty \nu(\overline{uv_x} - \overline{uu_y})\,dy + \int_0^\infty \overline{vu}\,dy - \underbrace{\int_0^\infty u(x,y)\,dy\int_y^\infty \frac{\partial}{\partial x}(u^2)\,dy'}_{I}$$

Consider $I = \int\int_0^\infty u(x,y)\,dy \int_y^\infty \frac{\partial}{\partial x}(u^2 - p)\,dy' \overset{\text{by parts x}}{=} \int\int v_y(x,y)\,dy\int_y^\infty (u^2 - p)\,dy'$

Reverse order $= \int\int (u^2 - p)\,dy' \int_0^{y'} v_y\,dy = \int \overline{(u^2 - p)v}\,dy \quad \therefore$ cancels previous term. $+\int \overline{pv}\,dy$

$$\text{Hence } U\nu\overline{f} = +\int_0^\infty \nu(\overline{v_y v} - \overline{uu_y})\,dy \;{}^{+\int \overline{pv}\,dy} = -\tfrac{1}{2}\nu(\overline{v^2} - \overline{u^2})\Big|_0^{\infty} {}^{+\int \overline{pv}\,dy} = +\tfrac{1}{2}\nu\overline{u_0^2} + \int \overline{pv}\,dy$$

$$\therefore \overline{U\,u(x,0) + \tfrac{1}{2}(u(x,0))^2} = 0 \qquad \therefore \boxed{\overline{(u(x,0) + U)^2} = U^2} + \int \overline{pv}\,dy/2\nu$$

$$\frac{1}{2\nu}\int \overline{pv}\,dy$$

The theorem is NOT exact!

$\nu \overline{U_y} = \overline{VU} \quad \therefore \nu \bar{f} = -\int_0^\infty VU \, dx\, dy$ $\quad V_0 = 0 \quad f = U_0 - U$

$$\nu \bar{f} = -\int_0^\infty dy \int_0^{2\pi} dx \int_y^\infty U_x(x, y')\, dy' \, U(x,y)$$

$\therefore$ consider $\int_y^\infty U_x(x,y')\,dy' = \int_y^\infty \omega$

$$\nu^2 U_{yyy} = \nu(U U_{xy} + U_{xy}) + V(UU_x) + V^2 U_y + U_x V$$

$$\tfrac{1}{2}\nu^2 (U^2_{yyy}) = 2\nu U(\cancel{U_y U_x} + U U_{xy} + U_{xy}) + V U^2 U_x + V^2 U U_y + U U_x V$$

$$\cancel{3\nu} U U_x U_y + 3\nu U_y V U_y + 3\nu U_y U_x$$

$$\nu^2\left(\tfrac{1}{2}U^2 + U\right)_{yyy} = \nu 3 U U_y U_x + \nu U^2 U_{xy} + 2\nu U U_{xy} \cancel{3\nu} U_y^2 V + 3\nu U_y U_x + \cancel{\nu U U_{xy}} + \cancel{U_x U_y} + \nu U_{xy}$$

$$+ V U^2 U_x + V^2 U U_y + 2 U U_x V + \cancel{V U U_x} + V^2 U_y + U_x V$$

$$= 3\nu (U U_y V)_y - 3\nu U U_{yy} V + 3\nu (U^2 U_x)_y - 2\nu U^2 U_{xy}$$

$+3$
-2
$+3$

$U \quad 2UU_y$

$\cancel{U^2 = U_{yyy} U +} \; 2U_{yy} U + 2U_y U_y$

$2U_{yyy} U + 6 U_{yy} U_y$

$U^2 + U$

$U^2 V_{yy} = \frac{d}{dy}(U^2 V_y)$

$\cancel{-2U_y U V_y}$

$-2\frac{d}{dy}(U_y U V)$

$+2 V U_y^2 - 2 V U U_{yy}$

~~Case 3 Circuit boundary~~

Case 2 Cont. unsymmetrical $\psi = (x^3 - 3y^2x)\beta + ax + cx^2$

$\therefore$ for $x>0$ $\quad x + cx + \beta(x^2 - 3y^2) + a = 0$

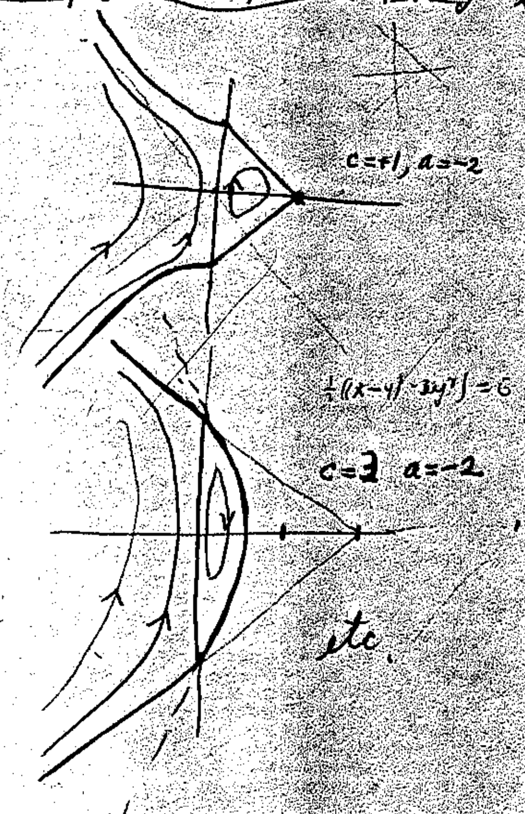
for $x<0$ $\quad -x + cx + \beta(x^2 - 3y^2) + a = 0$

~~Example~~ Eg. $\beta = -1/2$ $\quad \cancel{\tfrac{1}{2}(x+1)^2}$ $\cancel{\tfrac{1}{2}((x-1)^2 - 3y^2) = a + c + 1 = 0 \quad a + c = -1 \quad \to x = 1 \pm \sqrt{3}\,y}$ $\quad x>0$

$\cancel{\tfrac{1}{2}((x+1)^2 - 3y^2) = a + c + 1}$

$x>0$ $\quad \tfrac{1}{2}\left((x-(1+c))^2 - 3y^2\right) = a + \tfrac{1}{2}(1+c)^2$

$x<0$ $\quad \tfrac{1}{2}\left((x+(1-c))^2 - 3y^2\right) = a + \tfrac{1}{2}(1-c)^2$



There are innumerable possibilities.

$$\nu \overline{u''_{yy}} = \frac{\partial}{\partial y}\, \overline{v^0 u' + v' u^0} \quad \cancel{\nu \overline{u'_y} = \int \overline{v^0 u'}}$$

$$\nu \overline{u''_y} = \overline{v^0 u'} + \overline{v' u^0} \qquad \nu f'' = -\int_0^\infty \overline{v^0 u'}\, dy$$

$$-\int_0^\infty \overline{v' u^0}\, dy.$$

$$\nu u'_{yy} - u'_x = u^0 u^0_x + v^0 u^0_y$$

$$\nu \overline{u_{yy} u} = \overline{v u u_y} = \frac{\partial}{\partial y}\left(\overline{\frac{v u^2}{2}}\right) - \cancel{\overline{v_y \frac{u^2}{2}}}$$

$$-\nu \overline{u_y^2} + \frac{\partial}{\partial y}\, \overline{\nu u_y u} = \frac{\partial}{\partial y}\left(\overline{\frac{v u^2}{2}}\right)$$

$$\nu \int_0^\infty \overline{u_y^2}\, dy = \overline{\nu u_y u}\Big|_0 \qquad \overline{\nu u_y u} = \overline{\frac{v u^2}{2}} + \int_y^\infty \nu \overline{u_y^2}$$

$$u_y = \int_y^{+\infty} (u^2)_x - vu$$

$$\overline{u u_y}$$

$$\int_\mu \overline{u u_y} = \overline{u \int_y^\infty (u^2 u)_x} - \int_\mu \overline{v u^2}$$

$$\downarrow$$

$$+\overline{v_y \left(\frac{u^2}{2} + u\right)}$$

$$\overline{v\left(\frac{u^2}{2} + u\right)}$$

$$n \int \overline{u^n u_y}\, dy = \int n \overline{u^n \int_y^\infty (u^2 + u)_x} - \int \overline{v u^{n+1} n}$$

$$n^2 u^{n-1} v_y \int$$

$$\nu \overline{u_{yy} u_y} = \overline{u u_x v_{yy}} + \overline{u_y (vu)_y} + \overline{u_{yy} v_y}$$

$$\cancel{\nu u_y - \int \overline{u_y u_x} = \int_y v - u_y^2 u_x}$$

$$V^0_y = -u^0_x = \nu u^0_{yyy}$$

$$V^0 = -\nu u^0_y + \nu u^0_y(x, 0)$$

$$\frac{\partial}{\partial x} u^0_y(x, 0) = u^0_{yx}(x, 0) = -V^0_{yy}(x, 0) = \nu u^0_{yyy}(x, 0).$$

$$\cancel{\int v^0(x,y)\, u'(x,y)\, dy\, dx}$$

$$+\iint \cancel{v^0(x,y) \int_y^\infty u^0_x} \qquad v(x,y) = \int_y^\infty u_x(x, y')\, dy'$$

$$+\int_0^\infty \int_y^\infty u^0_x(x, y') \int_0^\infty u'(x, y)\, dx\, dy\, dy'$$

$$\iint v u\, dy\, dx$$

$$= \iiint_y^\infty u_x(x, y')\, u(x, y)\, dy\, dy'\, dx$$

$$= -\iiint_y^\infty u(x, y')\, u_x(x, y)\, dy\, dy'\, dx$$

$$= -\iiint_0^y u(x, y)\, u_x(x, y)\, dy\, dy'\, dx$$

$$= -\iiint_0^y u(x, y)\, u_{yy}(x, y')\, dy\, dy'\, dx$$

$$= -\iint u(x, y)\, [$$

$$\cancel{\iint v u\, dy\, dx = \iiint_0^y u(x, y)\, u_{yy}(x, y')\, dy\, dy'\, dx}$$

$$\cancel{= -\nu \iint u(x, y)}$$

$$\iint v u\, dy\, dx = \mp \iiint_0^y u_x(x, y')\, dy'\, u(x, y)\, dx\, dy$$

$$= +\iiint_0^y u(x, y')\, u_x(x, y)\, dx\, dy$$

$$= \iiint_y^\infty u_x(x, y')\, dy'\, u(x, y)\, dx\, dy$$

$$u_x = \nu u_{yy} + u u_x + v u_y = \nu u_{yy} + \frac{\partial}{\partial y}\left(\frac{v}{u}\right) \cdot u^2$$

$$\int_y^\infty u_x\, dy' = -\nu u_y - \int_y^\infty u^2\, d\!\left(\frac{v}{u}\right)$$

$$= -\nu u_y + (uv)_y + 2\int_y^\infty v u_y\, dy$$

$\nu U_{yy} = uU_x + VU_y - V_y$

$$V = \int_0^y \left(\nu u_{yy} \cancel{- uu_x} + 2Vu_y\right) \quad + uV_y + Vu_y$$

$\nu \overline{u_y} = \overline{Vu} =$

$$= \nu u_y + uV - 2\int_0^y V u_y \, dy$$

$$\overline{Vu} = \overline{V \nu u_y}$$

$\overline{\nu u_y} = \overline{Vu}$

$$\int_0^\infty dy\, u(x,y) \int_0^y u_x(x,y')\, dy'\, dx = 0?$$

$\cancel{\int_0^\infty u(x,y)\,dy}$

$\int_0^\infty u(x,y)\,dy = 0$ — Proved NOT TRUE.

$\int_0^\infty u_x \, dy = -V(\infty)$ $\overline{V_\infty} = 0.$

$$\int_0^\infty dy\, u(x,y) = A(x)$$

$$\int_0^\infty dy\, u_x(x,y) = -V(\infty) \quad \therefore \quad A_x(x) = -V(\infty).$$

$$\int_0^\infty dy\, u(x,y) \int_0^y dy'\, u_x(x,y')\, dx = \oint A_x A \quad \therefore \quad \overline{\cdots} = 0.$$

$$\delta H = -\frac{N}{T}\left\{(\Delta'+\mu h')\,\delta\rho_R - (\mu h'_L)\,\delta\rho_L\right\}(T\ln x) \qquad + \underline{M h'(\rho_L)\,\delta\rho_L}$$

$$+ \frac{\cancel{N}(T\ln xe)\ \rho_L}{M\cancel{- \frac{N}{V}\left(\frac{\mu}{\rho_R}-\frac{\mu}{\rho_L}\right)}}\left[-\frac{M}{\rho_L^2}\delta\rho_L - N\mu\left(\frac{\delta\rho_R}{\rho_R^2}-\frac{\delta\rho_L}{\rho_L^2}\right) - \frac{N}{T}\left(\frac{\mu}{\rho_R}-\frac{\mu}{\rho_L}\right)\left\{(\Delta'+\mu h')\,\delta\rho_R - (\mu h'_L)\,\delta\rho_L\right\}\right]$$

① $\delta\rho_R$ coef $=0$.

$$-\frac{N}{T}\left((\Delta'+\mu h')\right)(T\ln x) + \frac{NT\ln xe}{V-N\left(\frac{\mu}{\rho_R}-\frac{\mu}{\rho_L}\right)}\left[-N\mu\frac{\delta\rho_R}{\rho_R^2} - \frac{N}{T}\left(\frac{\mu}{\rho_R}-\frac{\mu}{\rho_L}\right)(\Delta'+\mu h')\right]=0$$

carry only 1st order: $-N\ln x(\Delta'+\mu h')=0 \quad \therefore \quad (\Delta'+\mu h')_R=0$ determines ρ_R.

② $\delta\rho_L$ coef $=0$:

$$+\frac{N}{T}(\mu h')(T\ln x) + \frac{NT\ln xe\,\rho_L}{M}\left[-\frac{M}{\rho_L^2} + \frac{N\mu}{\rho_L^2} + \frac{N}{T}\left(\frac{\mu}{\rho_R}-\frac{\mu}{\rho_L}\right)\mu h'_L\right] + Mh' = 0$$

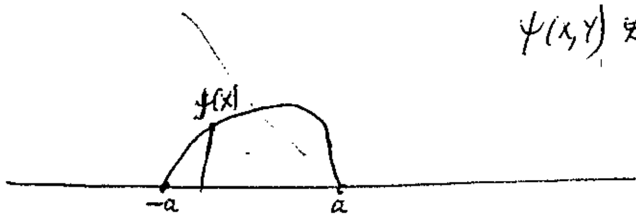
carrying only 1st order.

$$Mh'(\rho_L) + N\left\{\mu h'\ln(x) - \frac{T}{\rho_L}\ln(xe)\right\} = 0$$

and the Vol eqn. Call ρ_0 = Mean density of fluid $= \frac{M}{V} = \dfrac{M}{\frac{M}{\rho_L}+N\left(\frac{\mu}{\rho_R}-\frac{\mu}{\rho_L}\right)}$

$$\rho_0 = \rho_L\left(1-\frac{N\mu}{M}\left(\frac{\rho_L}{\rho_R}-1\right)\right)$$

$$\psi(x,y) = y + \omega \iint_R \ln\left(\frac{(x-x_2)^2+(y-y_2)^2}{(x-x_2)^2+(y+y_2)^2}\right) dx_2\, dy_2$$



Now, on the boundary of the region ψ must be zero.

Hence (if monotone?) $f(x) = \omega \int d\xi \int_0^{f(\xi)} d\eta \, \ln\left(\frac{(x-\xi)^2+(f(x)-\eta)^2}{(x-\xi)^2+(f(x)+\eta)^2}\right)$

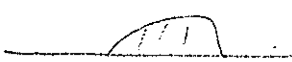
$$\int_0^f d\eta \ln\left((x-\xi)^2+(f-\eta)^2\right) = \int_0^f d\eta' \ln\left((x-\xi)^2+\eta'^2\right)$$

$$\int_0^f d\eta \ln\left((x-\xi)^2+(f+\eta)^2\right) \equiv \int_f^{2f} d\eta' \ln\left((x-\xi)^2+\eta'^2\right)$$

$$1-2 = \eta' \ln\left((x-\xi)^2+\eta'^2\right) - 2\eta' + 2(x-\xi)\tan^{-1}\frac{\eta}{x-\xi} \Big|_0^f - \Big|_f^{2f} \quad \therefore (0) - 2(f) + (2f)$$

$$= -2f \ln\left((x-\xi)^2+f^2\right) + 2f \ln\left((x-\xi)^2+4f^2\right) - 2(x-\xi)\left[2\tan^{-1}\frac{f}{x-\xi} - \tan^{-1}\frac{2f}{x-\xi}\right]$$

Notice:



OUTSIDE u, v have zero curl $\therefore$ exists a φ (OUTSIDE)

$$\cancel{\psi(x,y) = \int \omega(x',y') \ln(r_{x'}}$$

$$\psi(1) = \int_R \omega(2) \ln(r_{21}^2)\, d\tau_2$$

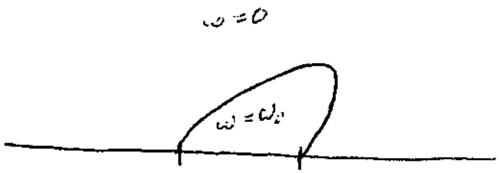
$$\frac{\partial \psi}{\partial x} = \int \omega(2) \frac{x_{21}}{r_{21}^2} d\tau_2 = -\frac{\partial \varphi}{\partial y}$$

$$\psi_y = \int \omega(2) \frac{y_{21}}{r_{21}^2} d\tau_2 = \frac{\partial \varphi}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\tan^{-1}\frac{x_{21}}{y_{21}}\right) = \frac{\frac{1}{y}}{1+\frac{x^2}{y^2}} = \frac{y}{x^2+y^2}$$

$$\varphi(1) = \int \omega(2) \tan^{-1}\left(\frac{x_{21}}{y_{21}}\right) d\tau_2 = \int s(2) \ln r_{21}^2\, d\tau_2$$

find s in terms of ω?



$\psi = \quad q = Ui + \sum \omega_c$

$= Ui + u,\ v$

$\omega = \dfrac{U_y - V_x}{}$; $U_x = -V_y$

$(q\cdot\nabla)q = \nabla p$

call $p + \frac{1}{2}(U^2+V^2) = \Pi$

$uU_x + VU_y$

$= uu_x + VV_x + V\omega$

$\nabla\Pi_x - V\omega = \nu\nabla^2 U$

$\nabla\Pi_y + U\omega = \nu\nabla^2 V$

Since V need not be const. put ~~dipole~~ ω_c source σ

$\therefore \omega$ = const inside

$= \delta(\text{surf})\, S(\theta)$ on surf

$= 0$ outside

$\nabla^2\psi = \omega$ $\therefore$ given $\omega \to \psi$

$U = \psi_y,\ V = -\psi_x$

ψ continuous ~~(arb const~~ (because ψ = const on line & can take equal const by adding arb const ψ)

$\frac{\partial\psi}{\partial n}$ not continuous (?) determined by $S(\theta)$

$\nabla^2\psi$ not cont – det by ω

The difficulty is to find a curve so that ψ turns out const on it.

$\cancel{\nabla p =}\ (q\cdot\nabla)(p + \tfrac{1}{2}q^2) = 0$

$\therefore (p + \tfrac{1}{2}q^2)_{INS} - (p + \tfrac{1}{2}q^2)_{OUT} =$

But

$\nabla p = (q\cdot\nabla)q$

$p_x = uU_x + VU_y = \psi_y\psi_{xy} - \psi_x\psi_{yy}$ $\cancel{= \psi_{xy}\psi_{xy} - \psi_x}$

$p_y = -\psi_y\psi_{xx} + \psi_y\psi_{yx}$

$\cancel{p_{xy} = \psi_y\psi_{xyy}}$

$p_x = \psi_y\psi_{xy} + \psi_x\omega + \psi_x\psi_{xx}$

$\Pi_x = \psi_x\omega$

$\Pi_y = \psi_y\omega$

$\therefore \Pi = \omega\psi + C_1$ INSIDE

$= C_2$ OUTSIDE.

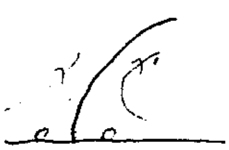
Choose $\psi = 0$ at edge

at inside $\Pi_1 = C_1 = p_1 + \frac{1}{2}q_1^2$

outside $\Pi_2 = C_2 = p_2 + \frac{1}{2}q_2^2$

But $p_1 = p_2$ at opposite pts

$\therefore \boxed{q_1^2 - q_2^2 = \text{const.}}$



But in the corner $q_1 = 0$, $q_2 = 0$ $\therefore$ const $= 0$

$\therefore q_1^2 = q_2^2$ and, as there are no sign reversals or the boundary will be unstable, $q_1 = q_2$ $\therefore \frac{\partial\psi}{\partial n}$ is continuous. $\underline{S = 0.}$

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$\cancel{\nu U_{xx} =}$ $\quad \nu U_{yy} = U U_x + V U_y \cancel{- \frac{\partial p}{\partial x}}$

$U_x = - V_y$

$\nu U_{xyy} = U_x U_x + U U_{xx} + V_x U_y + V U_{yx}$

$-\nu V_{yyy} = V_y V_y + U V_{yx} + V_x U_y - V V_{yy}$

$$\nu \nabla^2 \nabla^2 \psi = \psi_x \nabla^2 \psi_y - \psi_y \nabla^2 \psi_x = \nabla \cdot (\psi_x \nabla \psi_y - \psi_y \nabla \psi_x)$$

$\nu (V_{yy} + V_{xx}) = U V_x + V V_y - \frac{\partial p}{\partial y}$

$0 \quad \cancel{\nu \nabla^2 U_x} = U_x U_x \cancel{+ U U_{xx}} + V_x U_y \cancel{+ V U_{xy}} - p_{xx}$

$\cancel{\nu \nabla^2 V_y} + U_y V_x \cancel{+ U V_{xy}} + V_y^2 \cancel{+ V V_{yy}} - p_{yy}$

$\nu U_{yy} = \frac{\partial}{\partial x}(U^2) + \frac{\partial}{\partial y}(VU)$

$\int_0^x [\nu U_{yy} + (VU)_y]\, dx = U^2(x,y) - U^2(0,y)$

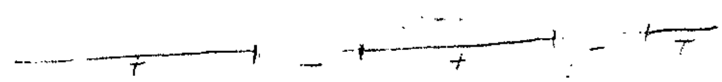
$\int_0^x [\nu U_y + VU]_0 \, dx = U^2(x,0) - U^2(0,0)$ $\cancel{- U^2(x,\infty) - U^2(0,\infty)}$

$x=0 \qquad x.$

$U = \psi_y \qquad V = -\psi_x$

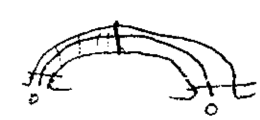
$\nu \psi_{yyy} = \psi_y \psi_{yx} - \psi_x \psi_{yy}$

Desire to prove $\oint \overline{\psi_y^2}\big|_0 - \overline{\psi_y^2}\big|_\infty = 0$



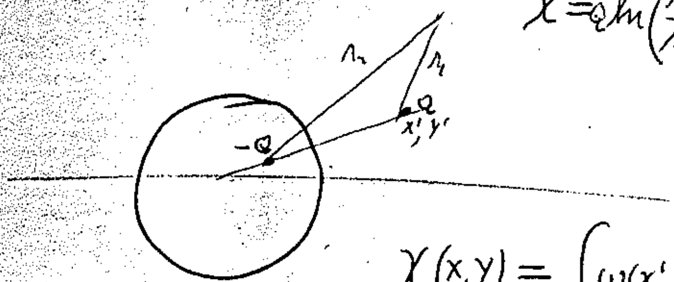
The pressure must be positive continuous across the free boundary.

$\nu \nabla^2 q = (q \cdot \nabla) q + \nabla p \qquad \therefore \nu\, q \cdot \nabla^2 q = (q \cdot \nabla)\left(\frac{q^2}{2} + p\right)$



Sense indicated $\quad \nu \int q \cdot \nabla^2 q \, dVol = \int (N \cdot q)\left(\frac{q^2}{2} + p\right) = q_t W\left(\frac{q_t^2}{2} + p_t\right)$

$cb = a^2$



$$\chi = Q \ln\left(\frac{r_1^2 b}{r_2^2 c}\right) \qquad \text{On circle } \frac{r_1^2}{r_2^2} = \left(\frac{c-a}{a-b}\right)^2 = \left(\frac{a^2-ab}{ab-b^2}\right)^2 = \frac{a^2}{b^2} = \frac{c}{b} = \frac{c^2}{a^2}$$

$$\chi(x,y) = \int \omega(x',y') \cdot \ln\left[\frac{\cancel{a^2}[(x-x')^2+(y-y')^2]\,(x'^2+y'^2)/a^2}{\left(x-\frac{a^2x'}{x'^2+y'^2}\right)^2 + \left(y-\frac{a^2y'}{x'^2+y'^2}\right)^2}\right] dx'dy'$$

$\frac{x_c}{y_c} = \frac{x'}{y'}$

$(x_c^2+y_c^2)(x'^2+y'^2) = a^4$

$x_c = \frac{x'}{\sqrt{x'^2+y'^2}} \cdot \frac{a^2}{\sqrt{x'^2+y'^2}} = \frac{a^2x'}{x'^2+y'^2}$

$r_2^2 = x^2+y^2 - \frac{2a^2(xx'+yy')}{x'^2+y'^2} + \frac{a^4}{x'^2+y'^2}$

$r_1^2 = x^2+y^2 - 2(xx'+yy') + (x'^2+y'^2)$

If x is on circle, so $x^2+y^2=a^2$, Then $\frac{r_2^2}{r_1^2} = \frac{a^2}{x'^2+y'^2}$

$\chi = 0$ on circle.

Desire, on circle $V_x = \frac{\partial\chi}{\partial y} = -\frac{2U}{a^2} y^2$

$V_y = -\frac{\partial \chi}{\partial x} = +\frac{2U}{a^2}xy$

1st condition: $V_x = \frac{\partial\chi}{\partial y} = \int 2\omega(x',y')\left[\frac{(y-y')}{r_1^2} - \frac{y - \frac{a^2y'}{x'^2+y'^2}}{r_2^2}\right]dx'dy'$

$V_y = -\frac{\partial\chi}{\partial x} = \int 2\omega(x',y')\left[-\frac{x-x'}{r_1^2} + \frac{x-\frac{a^2x'}{x'^2+y'^2}}{r_2^2}\right]dx'dy'$

On cylinder, $r_2^2 = r_1^2\frac{a^2}{x'^2+y'^2}$

$V_x = y\int 2\omega(x',y')\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right]dx'dy'$

$V_y = -x$ ditto.

$\therefore$ Condition for solution

$$-\frac{2U}{a^2}y = \int 2\omega(x',y')\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right]_{\text{circle}} dx'dy'$$

$$-\frac{2U}{a^2}y = \iint_{\text{outside circle}} 2\omega(x',y')\left[1-\frac{x'^2+y'^2}{a^2}\right]\frac{dx'dy'}{a^2-2(xx'+yy')+x'^2+y'^2}$$

$(x^2+y^2=a^2)$

$$\varphi = UY - \frac{Ua^2Y}{X^2+Y^2}$$

$$\psi = UX + \frac{Ua^2X}{X^2+Y^2}$$

$$\bar{V}_x = \frac{\partial\varphi}{\partial Y} = U + \frac{Ua^2(Y^2-X^2)}{(X^2+Y^2)^2} = \frac{\partial\psi}{\partial x}$$

$$\bar{V}_y = -\frac{\partial\varphi}{\partial x} = -\frac{2Ua^2XY}{(X^2+Y^2)^2} = \frac{\partial\psi}{\partial Y}$$

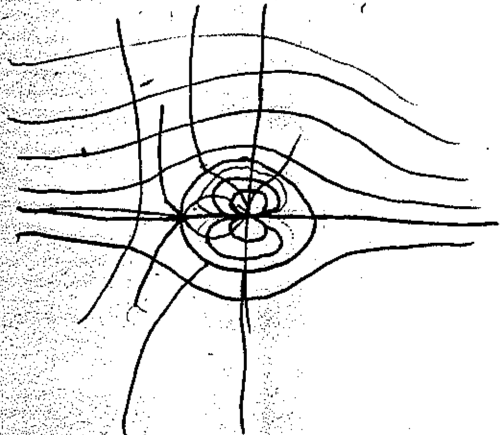
$$\omega_{\varphi\varphi} + \omega_{\psi\psi} = \frac{1}{\mu}\omega_\psi$$

$$\omega = e^{\frac{\psi}{2\mu}} g$$

$$g_{\varphi\varphi} + g_{\psi\psi} = \frac{1}{4\mu^2} g$$ Let a=1.

$$\varphi = U(\rho - \frac{a^2}{\rho})\sin\alpha.$$

$$\psi = U(\rho + \frac{a^2}{\rho})\cos\alpha$$



$$\frac{\partial V_x}{\partial Y} - \frac{\partial V_y}{\partial X} = \omega$$

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0$$

$V_x = V_y = 0$ at circle.

$$\nabla^2\chi = \omega \qquad V_x = \frac{\partial\chi}{\partial Y},\ V_y = -\frac{\partial\chi}{\partial X}$$

$\chi = 0$ on circle

$\frac{\partial\chi}{\partial n} = 0$ on circle

consider $\int f\omega\, dVol = \int f(\nabla\times V)dVol = -\int V\times(\nabla f)dVol + \int (N\times V)f\, dSurf$ ~~Suppose~~ $\cancel{\nabla\chi}$

$\cancel{\int a\omega\, dVol = \int a\times(\nabla\times V)dVol}$

$$\nabla^2 V_x = \frac{\partial\omega}{\partial Y}$$

$$\nabla^2 V_y = -\frac{\partial\omega}{\partial X}$$

$$\int \left(g_\varphi^2 + g_\psi^2 + \frac{1}{4\mu^2}g^2\right)dVol = Min$$

$\int$

$\cancel{\int \omega_\varphi^2 + \omega_\psi^2 + \frac{1}{\mu}\omega\omega_\psi}$

φ, α. Inversions: $\alpha \to -\alpha$, $\rho \to a^2/\rho$. $\therefore \varphi \to -\varphi$, $\psi \to \psi$

φ

$$\varphi^2+\psi^2 = U^2(x^2+y^2)\left(1 + \frac{a^2(y^2-x^2)}{(x^2+y^2)^2} + \frac{a^4}{(x^2+y^2)^2}\right)$$

$-2Ua$ $\quad$ $2Ua$ $\quad$ ψ

$$\omega(\varphi,\psi) = \frac{\partial}{\partial\psi}\left[\int_{-2Ua}^{+2Ua} F(\psi')\, I_0\left(\sqrt{\varphi^2+(\psi-\psi')^2}\right)d\psi'\right]$$

F unknown.

$$\chi(x,y) = \int \omega(x',y')\ln\left(\cancel{\sqrt{x'^2+y'^2}}\sqrt{(x-x')^2+(y-y')^2}\right)dx'dy'$$

$$V_x = \int \omega(x',y')\frac{(y-y')}{(x-x')^2+(y-y')^2}dx'dy' + U$$

$V_y = \int \cdots (x-x')$

Desire, at surface V is tangential, of value $-2U\sin\alpha$.

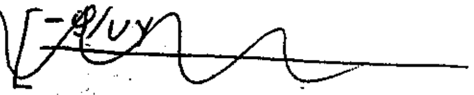
Consider $-\frac{2U}{a}a\sin\beta = \iint \omega(X',Y')\left[\frac{a^2-(X'^2+Y'^2)}{a^2+(X'^2+Y'^2)-2a(X'\cos\beta+Y'\sin\beta)}\right]dX'dY'$

To try express X', Y' in terms of φ, χ

$$\cancel{dxdy =} \; d\varphi d\psi = \begin{vmatrix} \frac{\partial\varphi}{\partial X} & \frac{\partial\psi}{\partial X} \\ \frac{\partial\varphi}{\partial Y} & \frac{\partial\psi}{\partial Y}\end{vmatrix} dX dY = (V_x^2+V_y^2)\,dX\,dY$$

$$V_x^2+V_y^2 = U^2 + \frac{2U^2a^2(Y^2-X^2)}{(X^2+Y^2)^2} + \frac{U^2a^4}{(X^2+Y^2)^2}$$

$$\varphi^2+\psi^2 = U^2R^2\left(1 \cancel{+} \frac{a^2(Y^2-X^2)}{R^4} + \frac{a^4}{R^4}\right)$$

$\omega(\varphi,\psi)$  φ

$U=1$

$$\boxed{V_x^2+V_y^2 = 2\left(1+\frac{a^4}{R^4}\right) - \frac{\varphi^2+\psi^2}{R^2}}$$

$$\frac{a^2-(X^2+Y^2)}{a^2+(X^2+Y^2)-2aX\cos\beta-2aY\sin\beta} = \frac{a^2-(X^2+Y^2)}{a^2+(X^2+Y^2)-\frac{2a\psi\cos\beta}{1+\frac{a^2}{X^2+Y^2}}-\frac{2a\varphi\sin\beta}{1-\frac{a^2}{X^2+Y^2}}}$$

clear fractions $\dfrac{(a^2-R^2)^2(a^2+R^2)}{(a^2+R^2)^2-2a\psi\cos\beta R^2(R^2-a^2)-2a\varphi\sin\beta R^2(a^2+R^2)}$

$$\frac{\varphi}{1-\frac{a^2}{R^2}} = Y \;;\; \frac{\psi}{1+\frac{a^2}{R^2}} = X \qquad \frac{R^4\varphi^2}{(R^2-a^2)^2}+\frac{R^4\psi^2}{(R^2+a^2)^2} = 1$$

$$(\varphi^2+\psi^2)(R^4+a^4)+(\varphi^2-\psi^2)(2a^2R^2) = \frac{1}{R^2}(R^8-2a^4R^4+a^8)$$

$$(\varphi^2+\psi^2)\left(\frac{R^2}{a^2}+\frac{a^2}{R^2}\right)+(\varphi^2-\psi^2)\cdot 2 = a^2\left(\left(\frac{R^2}{a^2}\right)^2-2+\left(\frac{a^2}{R^2}\right)^2\right)$$

$$\frac{R^2}{a^2}+\frac{a^2}{R^2} = T$$

$$\left(\frac{R^2}{a^2}\right)^2 - T\left(\frac{R^2}{a^2}\right)+1=0$$

$$\frac{R^2}{a^2} = \frac{T}{2}\pm\sqrt{\frac{T^2}{4}-1}$$

$$(\varphi^2+\psi^2)T+2(\varphi^2-\psi^2) = -4a^2+a^2T^2$$

$$T^2 - \frac{\varphi^2+\psi^2}{a^2}T - \frac{4a^2+2(\varphi^2-\psi^2)}{a^2} = 0$$

$$T = \frac{\varphi^2+\psi^2}{2a^2}\pm\sqrt{\left(\frac{\varphi^2+\psi^2}{2a^2}\right)^2+\frac{4a^2+2(\varphi^2-\psi^2)}{a^2}}$$

$\varphi = 2a\alpha$

$\psi = 2a\beta$

$$4\alpha^4+8\alpha^2\beta^2+4\beta^4+4+8\alpha^2-8\beta^2$$

$$4\left[(\alpha^2+1)^2+(\beta^2-1)^2+2\alpha^2\beta^2\right]$$

$$T^2-4 = \left(\frac{\varphi^2+\psi^2}{a^2}\right)^2\frac{1}{2}+2\frac{\varphi^2-\psi^2}{a^2}+\frac{\varphi^2+\psi^2}{a^2}\sqrt{\left(\frac{\varphi^2+\psi^2}{a}\right)^2\frac{1}{4}+\frac{2(\varphi^2-\psi^2)}{a^2}+4}$$

$$= \frac{1}{2}\left(\frac{\varphi^2+\psi^2}{a^2}\right)^2\left[1+4\frac{(\varphi^2-\psi^2)a^2}{(\varphi^2+\psi^2)^2}+\sqrt{1+16\frac{a^4}{(\varphi^2+\psi^2)^2}+8\frac{(\varphi^2-\psi^2)a^2}{(\varphi^2+\psi^2)^2}}\right]$$

$$\frac{-(a^2-R^2)}{\left[(a^2+R^2)-\frac{2a\psi\cos\beta}{1+\frac{a^2}{R^2}}-\frac{2a\varphi\sin\beta}{1-\frac{a^2}{R^2}}\right]}\left(\cancel{2}\sqrt{V_x^2+V_y^2}\right.$$

$$\varphi^2-\psi^2=\left(1+\frac{a^4}{R^4}\right)(Y^2-X^2)-2a^2$$

$$\varphi^2+\psi^2=R^2\left(1+\frac{a^4}{R^4}-\frac{2a^2}{R^4}(Y^2-X^2)\right)$$

$$V_x^2+V_y^2=1+\frac{a^4}{R^4}+\frac{2a^2}{R^4}(Y^2-X^2)$$

$$=1+\frac{a^4}{R^4}+\frac{2a^2}{R^4}\left(\frac{\varphi^2-\psi^2+2a^2}{1+\frac{a^4}{R^4}}\right)$$

$$=\frac{\frac{R}{a}\sqrt{T-2}}{\left[\frac{R}{a}\left[\sqrt{T+2}-\frac{2a\psi\cos\beta}{\sqrt{T+2}}-\frac{2a\varphi\sin\beta}{\sqrt{T-2}}\right]\right]}\Big\{$$

$$\frac{R}{a}+\frac{a}{R}=\cancel{\sqrt{T+4}}\sqrt{T+2}$$

$$\frac{R}{a}-\frac{a}{R}=\sqrt{T-2}$$

$$\frac{R^2}{a^2}+\frac{a^2}{R^2}=T$$

$$V_x^2+V_y^2=$$

$$g(\varphi,\psi)=\frac{\partial}{\partial\varphi}\oint\frac{\mathcal{S}(\theta)}{\cancel{\psi(\alpha)\,d\alpha}}\oint e^{-\varphi\cos\gamma-(\psi-\psi)\sin\gamma\;\; 2a\cos\theta}\,d\gamma\,d\theta\,e^{\psi}\qquad \omega=e^{\psi}g$$

$$=\oint\oint S(\theta)\cos\gamma\,e^{-\varphi\cos\gamma-\psi(\sin\gamma-1)-a\cos\theta\sin\gamma}\,d\gamma\,d\theta$$

$$\omega(\rho,\alpha)=\oint\oint S(\theta)\cos\gamma\,\exp\left(-\left(\rho-\frac{a^2}{\rho}\right)\sin\alpha\cos\gamma-\left(\rho+\frac{a^2}{\rho}\right)\cos\alpha(\sin\gamma-1)-a\cos\theta\sin\gamma\right)d\gamma\,d\theta$$

$$\text{and }-2va\sin\beta=\oint\int_a^\infty\rho\,d\rho\,d\alpha\,\omega(\rho,\alpha)\left[\frac{a^2-\rho^2}{a^2+\rho^2-2a\rho\cos(\beta-\alpha)}\right]$$

$$-2Va\sin\beta=\oint\int\oint\oint S(\theta)\sin\gamma\,\rho\,d\rho\,d\alpha\,d\theta\,d\gamma\left[\frac{a^2-\rho^2}{a^2+\rho^2-2a\rho\cos(\beta-\alpha)}\right]\cancel{\exp}\left[-\rho\sin(\alpha+\gamma)+\frac{a^2}{\rho}\sin(\gamma-\alpha)-a\cos\theta\sin\gamma\right]$$

$$\cdot\exp\left(-\left(\rho-\frac{a^2}{\rho}\right)\sin\alpha\sin\gamma-\left(\rho+\frac{a^2}{\rho}\right)\cos\alpha(\cos\gamma-1)-a\cos\theta\cos\gamma\right)$$

$$\iiint_0^\infty S(\theta)\sin\gamma\,e^{2u}\,du\left[\frac{\sinh u}{\cosh u-\cos(\beta-\alpha)}\right]e^{-\left(2\sinh u\sin\alpha\sin\gamma+2\cosh u\cos\alpha(\cos\gamma-1)+a\cos\theta\cos\gamma\right)}$$

$(\nu^{1/2} L_0^{1/2} U_0^{-1/2})\, \psi_{yyyy} = \psi_{yyy}\psi_x - \psi_y \psi_{xyy}$ | $\psi = \nu^{1/2} L_0^{1/2} U_0^{1/2} \psi''$

$$\psi''_{yyyy} = \psi''_{yyy}\psi''_x - \psi''_y \psi''_{xyy}$$

$\cancel{\psi \sim \psi'}$ $\quad \dfrac{\partial}{\partial y} \to \nu^{\alpha} \dfrac{\partial}{\partial y'} \qquad \psi \to \nu^{-\alpha}\psi'$

$$\frac{\partial}{\partial x} \to \nu^{\beta} \frac{\partial}{\partial x'}$$

$$\nu \nu^{4\alpha} \nu^{-\alpha} \psi_{yyyy} = \nu^{\beta} \nu^{3\alpha} \nu^{-2\alpha} (\psi_{yyy}\psi_x - \psi_y \psi_{xyy})$$

$\nu^{1+3\alpha} = \nu^{\alpha+\beta} \qquad 1+2\alpha = \beta \qquad \alpha > 0 \therefore \beta > 1.$

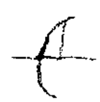
$x' = \nu^{\beta} x$ $\quad$ $\cancel{\dfrac{y^2}{x} = \dfrac{\nu^{2\alpha}}{\nu^{\beta}}\dfrac{y'^2}{x'} = \dfrac{1}{\nu}\dfrac{y'^2}{x'} \quad \therefore \dfrac{y'^2}{x'} = \nu \dfrac{y^2}{x} \to 0}$

$y' = \nu^{\alpha} y$

Eg' $\alpha = \frac{1}{3}$, $\beta = \frac{5}{3}$

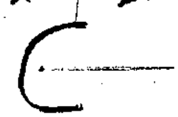
$\alpha = \frac{1}{2}$, $\beta = 2$

$$\frac{y'^2}{x'} = \frac{\nu^{\alpha}}{\nu^{\beta}}\frac{y^2}{x} = \frac{1}{\nu}\frac{y^2}{x} \to \infty.$$



$$\frac{y'^4}{x'} = \frac{\nu^{4\alpha}}{\nu^{\beta}}\frac{y^4}{x} = \text{ins}$$

Necessarily flatter than parabola.



$$\psi_{yyyy} = \psi_{yy}\psi_x - \psi_y \psi_{xy} + f(x)$$

$$-u_{yy} = -V u_y - u u_x + f(x)$$

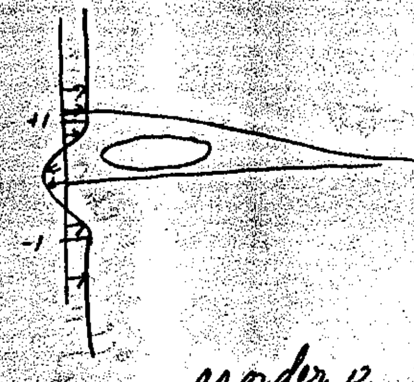
far away, $f(x) + \frac{1}{2}(u^2)_x = 0$?

$\psi - y + \delta(x)$?

Center li

LONG WAVE II.

$v = \psi_x$
$u = -\psi_y$

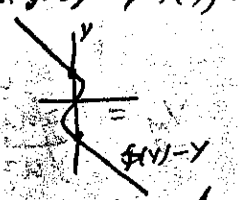


Suppose we have an input velocity distribution, ~~supposed~~ supposed that $v=0$ at $x=0$ but $u = f(y)$ as shown.

What happens for small ν.

Arguments indicate that as $\nu \to 0$, things get longer as order ν. $\therefore$ variables $x \sim \frac{1}{\nu}$, $y \sim 1$, $\psi \sim 1$. $u \sim 1$, $v \sim \nu$.

$\psi_0\left(\frac{0,Y}{}\right) = f(Y) - Y$



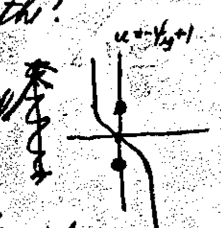
Equ. is $\boxed{\nu \nabla^2(\nabla^2\psi) = \psi_x \nabla^2\psi_y - \psi_y \nabla^2\psi_x}$ Exact.

Now $\frac{\partial}{\partial x} \sim \nu$ $\therefore$ $\nu \psi_{yyyy} = \psi_x \psi_{yyy} - \psi_y \psi_{xyy}$. just as in boundary layer theory.

$\therefore \nu \psi_{yyy} = \psi_x \psi_{yy} - \psi_y \psi_{xy} + \nu F(x\nu)$ (integration constant is order ν. ~~as $\nu \to 0$ it is zero?~~)

Now let us remove the uniform flow, $\psi = y$. $\therefore$ Call $\psi = y + \psi'$ but leave out the '.

$\nu \psi_{yyy} = \psi_x \psi_{yy} - \psi_y \psi_{xy} + \psi_{xy} + \nu F(x\nu)$ Initial $\psi(0,Y) = f(Y)$



The line $y=0$, is a streamline by symmetry $\therefore$ take $\psi(x,0) = 0$ &

Also assume that $y \to -y$, $\psi \to -\psi$ $\therefore$ Symmetry does not make $F(x) = 0$ so we must retain it.

Put $x\nu = X$.

$\boxed{\psi_{yyy} = \psi_x \psi_{yy} - \psi_y \psi_{xy} + \psi_{xy} \cancel{+F(X)}}$

Now if at any finite x we go very far away in y, ψ is nearly constant $\therefore$ $F(x) = 0$ (?)

OTHERWISE THERE IS A DISPLACED LAYER: I THINK PUTTING $F=0$ IS WRONG.

~~Let the excess velocity~~

Method of solution: ~~Negl~~

~~1st~~ 1st order Neglect quad. terms and solve

2nd order Put in 1st order quad. terms & solve. etc.

This Method is No Good.

1st order ψ, write as Ψ $\Psi_{yyy} = \Psi_{xy}$; Integrating: $\Psi_{yy} = \Psi_x$ ~~$+ G(X)$~~ ~~$\therefore \Psi(x,y) = \frac{1}{\,} \int e^{-y^2}$~~

$\therefore \Psi(x,y) = \int \frac{e^{-\frac{(y-y')^2}{4\pi x}}}{\sqrt{4\pi x}} \cdot f(y')\,dy'$ $= \Psi(x=0)$.

$\Psi_x \Psi_{yy} - \Psi_y \Psi_{xy} = \Psi_{yy}^2 - \Psi_y \Psi_{yyy}$

NOTICE. IF THE VELOCITY IN x DIRECTION u AT SURFACE IS SPECIFIED, THEN THE DIFF. EQU. DETERMINES V INCLUDING. ~~IF V IS ALSO~~ AT SURFACE $U_{yy} = \cancel{Vu_y + V_y u}\; Vu_y - uV_x + V_y$. ; ie $V(0,Y) = U(0,Y) \int_0^Y \frac{U u_{yy}}{[U(0,Y)]^2}\,dy$

$$u_{yy} = V u_y - u V_y$$

$$\frac{1}{u}\int^y \frac{u_{yy}}{u^2}\,dy' = (V\ldots)$$

$$= u\left[\frac{u_y}{u^2} + \int^y \frac{2u_y^2}{u^3}\,dy'\right]$$

$$V_y = \int^y \frac{u_{yy}}{u^2} + \frac{u_{yy}}{u_y u} = \frac{u_x}{u_y}$$

$$\frac{\partial}{\partial y}\left(\frac{u_x}{u_y} - \frac{u_{yy}}{u_y u}\right) = \frac{u_{yy}}{u^2}$$

$$\frac{u_{xy}}{u_y}$$

$$S_{yy} + V(y)S = 0$$

$$T_{yy} + V(y)T = 0$$

$$S_{yy}T - T_{yy}S = 0 = \frac{\partial}{\partial y}(S_yT - T_yS)$$

$$\ln C = D$$

$$D_x = E - D_y\int_0^y E\,dy$$

$$D = \ln u$$

$$u u_x = u_{yy} - u u_y \int_0^y \frac{u_{yy}}{u^2}\,dy$$

$$\left(\frac{u_y}{u}\right)_y = \frac{u_{yy}}{u} - \frac{u_y^2}{u^2}$$

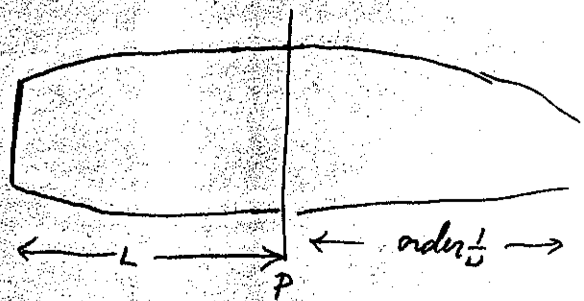
$$u u_x = u_{yy} - \frac{u_y^2}{u} + 2u_y\int_0^y \frac{u_y^2}{u^3}\,dy$$

$$u_x = \frac{\partial}{\partial y}\left(\frac{u_y}{u}\right) + 2\left(\frac{u_y}{u}\right)\int_0^y \left(\frac{u_y}{u}\right)^2 \frac{dy}{u}$$

$$\frac{\partial}{\partial y}\left(X\int_0^y \frac{q^2}{u}\right) = \frac{\partial X}{\partial y}\int + X\frac{q^2}{u}$$

$$\frac{2q}{q_y} = \frac{Xyu}{Xq^2}$$

~~CAN WE F~~ CAN WE FIND POINT TO DIVIDE WAKE?



The problem is this:

If we assume that the wake stretches for a distance of order $1/\nu$, we can set up ~~boundary~~ equations beyond the plane P, which ~~have~~ are consistent with this hypothesis, and have the property: That given u distribution, the subsequent v distribution required is determined ~~(are you crazy?)~~ by the equations in terms of $\&\ u$, — and v is of order ν.

Next, we wish to be able to choose a distance L so large that the v ~~not~~ will be of order ν, ~~if~~ and _yet_ we want viscosity not to be important previous to L. Can we do it?

Case 1. Suppose wake has finite width. Then the inclinations of the lines of flow are of order $\frac{h}{L}$ (where $h=1=$ height of barrier), and the v are therefore of order $\frac{1}{L}$. To keep this of order ν, L must be of order $\frac{1}{\nu}$,

So we cannot divide the regions.

The Interior Flow, for ν very small.

Suppose, as we have shown in other places, the flow is $\omega = \text{const}$ in regions except for two kinds of discontinuities:

1. Internal boundaries. ω jumps. Here we have (probably) shown that u is continuous, ω is <u>discont.</u>
   p is continuous

← surface

and if we take $\psi = 0$ on all the boundaries (all at once, of course) then R is continuous, where $R = p + \frac{1}{2}(u^2+v^2) - \omega\psi$.

$\omega = v_x - u_y = \nabla^2\psi$
$u = -\psi_y$
$v = \psi_x$
$u_x = -v_y$

2. Surface boundary layers. u-jumps. Here we have shown that p is continuous
   $q_n \sim 0$ ($q_n = 0$ at surface)

   R jumps, but $\Delta R = +\frac{1}{2}(\Delta u^2)$ (in general $\Delta R = +\frac{1}{2}\Delta q_t^2$)

q_n = q normal
q_t = q tangential to a surface

We call the actual surface velocity f.
We call the ideal surface velocity which eliminates surface boundary layers, (but leaves internal boundaries unchanged) g. Then ~~ΔR~~ $R_{\text{just inside}} - R_{\text{surface}}$
$= +\frac{1}{2}(g^2 - f^2)$.

Now the equations of motion, exactly, are

$$R_x = -\nu\omega_y - \omega_x\psi$$
$$R_y = \nu\omega_x - \omega_y\psi$$

Therefore inside each region where ω is constant, or nearly so, R is constant.
But from 1., R is continuous across surface boundaries.
Hence <u>R is the same at all points in the interior</u> (almost exactly for small ν).

Therefore, just inside the surface boundary layer R is constant $= C$.
(Hence: The mean R at the surface $= C \cancel{+} -\frac{1}{2}(\overline{g^2 - f^2})$. $= \bar{p}_s + \frac{1}{2}\overline{f_s^2}$.

($\bar{p}_s = C - \frac{1}{2}\overline{g^2}$ doesn't tell anything, just says $\bar{p}_s = \bar{p}_{\text{just inside}}$)

INTEGRAL THEOREMS   NOT INVOLVING PRESSURE.

$$\nu \nabla^2 \omega = v\omega_y + u\omega_x = \nabla\cdot(q\omega)$$

$v = \psi_x$
$u = -\psi_y$. $\omega = v_x - u_y$

Take $\psi = 0$ at boundary.

ψ = const = stream line.

Take any function of ψ such as $F(\psi)$, $F(0)=0$ at boundary.

$\nabla\psi$ = mag of velocity, but directed $\perp$ to it.
$\therefore \nabla\psi\cdot q = 0$.

$$\nu F(\psi)\nabla^2\omega = F(\psi)\nabla\cdot(q\omega)$$

$$\nu\nabla\cdot(F(\psi)\nabla\omega) - \nu F'(\psi)\nabla\psi\cdot\nabla\omega = \nabla\cdot(F(\psi)q\omega) - \cancel{F'(\psi)(\nabla\psi\cdot q)\omega}$$

Now if we integrate over the entire region, since the outside is a streamline, $F(\psi)=0$ there, and the surface integrals vanish! Call $F'(\psi)=f(\psi)$

$\therefore \int f(\psi)\underbrace{\nabla\psi\cdot\nabla\omega}\, dVol = 0$ for any $f(\psi)$ $\therefore$ $\boxed{\oint_{str.\, line}\frac{\partial\omega}{\partial n}ds = 0}$ as we know.

Eg. $f(\psi)=1$. $\nabla\psi\cdot\nabla\omega = v\omega_x - u\omega_y$. $\therefore \int(v\omega_x - u\omega_y)\,dVol = 0$.

also $= \frac{\partial}{\partial x}(v\omega) - \frac{\partial}{\partial y}(u\omega) + (v_x - u_y)\omega$ $\therefore \int(v\omega N_x - u\omega N_y)\,dS_{surf} + \int\omega^2 dVol = 0$.

$\therefore$ $\boxed{\oint_{SURF}\omega q_t\, ds + \int\omega^2 dVol = 0.}$

This might be useful for barriers??

~~Eg $f(\psi)=\psi$. $\int(\psi v\omega_x - \psi u\omega_y)\,dVol = 0$~~

~~$\psi v\omega_x - \psi u\omega_y = \frac{\partial}{\partial x}(\psi v\omega)$~~

$$f(\psi)v\omega_x - u\omega_y f(\psi) = \frac{\partial}{\partial x}(v\omega f) - \frac{\partial}{\partial y}(u\omega f) - f'(\psi)(v^2+u^2)\omega \quad \overset{}{-f\omega^2}$$

suppose $f(0)=0$.

$\therefore$ ~~$\int g(\psi)(v^2+u^2)\omega\, dVol = 0$ any $f(\psi)$ This is impossible~~

$$\int f'(\psi)(v^2+u^2)\omega\, dVol + \int f(\psi)\,\omega^2 dVol = 0.$$

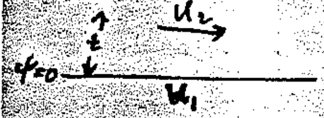
Example, $f=\psi$.

$$\int\psi\omega^2 dVol = 2\int\omega(u^2+v^2)\,dVol.$$

For any Surface $\oint_{surf}[\nu(\psi\nabla\omega - \omega\nabla\psi)\cdot N - q\psi\omega\cdot N]\,dS_{surf} = -\nu\int\omega^2 dVol.$

$\nabla\psi\cdot N = q_t$; $N\cdot q = q_n$.

Now we see if R is continuous across a velocity boundary layer.



$R_y = \nu\omega_x - \omega_y\psi$.

$\omega = v_x - u_y$

$u_x = -v_y$

$R_2 - R_1 = \nu\int_1^2 \omega_x\,dy - \omega\psi\big|_2 - \int_1^2 \omega u\,dy$

(a) $\int_1^2 \omega_x\,dy = \int_1^2 (v_{xx} - u_{yx})\,dy \sim \underset{\text{Negl.}}{\epsilon v_{xx}} - u_x\big|_1^2 = u_{x2} - u_{x1}$ is finite $\therefore \nu\int\omega_x dy$ is Negl.

(b) $\omega_2\psi_2$, ω_2 is finite but $\psi_2 = \int_1^2 \psi_y\,dy = -\int_1^2 u\,dy$ But u lies between u_1 & u_2 & is finite, hence $\int_1^2 u\,dy \sim \epsilon u \to 0$.

(c) $\int_1^2 \omega u\,dy. = \int_1^2 (v_x u - u_y u)\,dy$ Now $v_x \sim \sqrt{\nu}$, and $u \sim 1$ $\therefore$ 1st term $=0$.

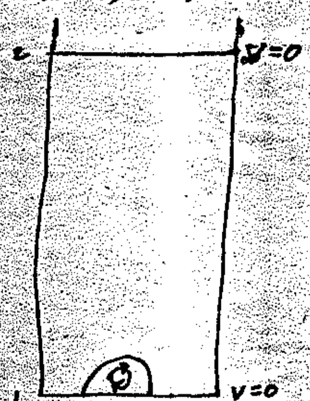
$\therefore -\frac{1}{2}(u_2^2 - u_1^2)$.

Hence $R_2 - R_1 = +\frac{1}{2}(u_2^2 - u_1^2)$. So R is not continuous, but jumps as u^2 jumps.

~~On the other hand let us figure the jump in R_x.~~

This essentially just says that of the terms $p + \frac{1}{2}u^2 + \frac{1}{2}v^2 - \omega\psi$, v^2 and $\omega\psi$ are nearly 0, & p is continuous

Overall exact theorems; for periodic cell.



$\overline{R_x} + \nu\overline{\omega_y} = -\overline{\omega_x\psi} = \overline{\omega v}$, the famous result. $\therefore \overline{\omega_{y1}} = \overline{\omega_{y2}}$

$\overline{R_y} = -\overline{\omega_y\psi} = -\overline{(\omega\psi)_y} - \overline{\omega u} = -\overline{(\omega\psi)_y} - \overline{v_x u} + \overline{u_y u}$

$= -\frac{\partial}{\partial y}\left(\omega\psi + \frac{1}{2}v^2 + \frac{u^2}{2}\right)$

Integrate $\overline{R} + \overline{\omega\psi} + \frac{1}{2}\overline{v^2} - \frac{1}{2}\overline{u^2} = \text{const.}$ $\therefore$ $\overline{p + v^2}$ = constant for every level.

$\therefore \overline{p_2} = \overline{p_1}$.

Let us write $u = -\psi_y$, $v = \psi_x$ $\therefore q_n = \frac{\partial\psi}{\partial t}$, $q_t = -\frac{\partial\psi}{\partial n}$. where n, t are normal & tangent to a line.

Then all our theorem. $\Pi_2 - \Pi_1 = \int_1^2 \left(q_n \omega + \nu \frac{\partial\omega}{\partial n}\right) ds = \int_1^2 \left(\frac{\partial\psi}{\partial s}\omega + \nu\frac{\partial\omega}{\partial n}\right) ds$

If 1, 2 lie in a region of constant ω, $\Pi_2 - \Pi_1 = \omega(\psi_2 - \psi_1)$.

So ~~$\Pi + \omega$~~ $R = \Pi - \omega\psi$ = const within a region of constant ω.

In general $\Pi_x = -\nu\omega_y + v\omega = -\nu\omega_y + \psi_x\omega$

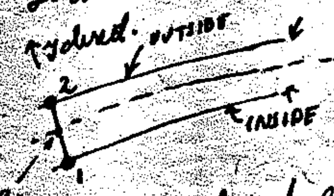
$\Pi_y = \nu\omega_x + \psi_y\omega$

$\therefore$ ~~Π~~ Call $\Pi - \omega\psi = R$ then

$R_x = -\nu\omega_y - \omega_x\psi$

$R_y = \nu\omega_x - \omega_y\psi$

We use these ~~two~~ show that R is continuous accross a free ω-jump boundry layer, ~~[This is clear now]~~ if ψ is zero inside the layer somewhere.



thickness $t \sim \nu^{1/3}$ (?)

$t \to 0$ as $\nu \to 0$.

Now let y be the direction of a normal

$\int_1^2 R_y\, dy = R_2 - R_1 = \nu\int_1^2 \omega_x\, dy - \int_1^2 \omega_y \psi\, dy$

$\int_1^2 \omega_x dy$ is between ~~$\int_1^2 \omega_{1x} dy$~~ $\omega_{1x} t$ and $\omega_{2x} t$, ~~because~~ and is multiplied by ν anyway, so is very small.

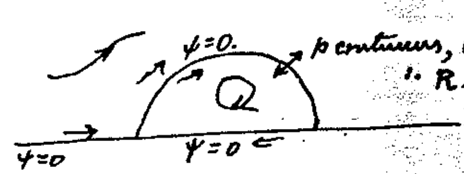
$\int_1^2 \omega_y \psi\, dy = \omega\psi\Big|^2 - \omega\psi\Big|^1 + \int_1^2 \omega u\, dy$

Now inside, $\bar\omega$ and $\bar u$ are both finite so last quantity is order t. $\psi_2 = \int_a^2 \psi_y dy \sim u_2 t$ $\therefore \psi_2 \sim \nu^{1/3}$

$\therefore \omega_2\psi_2 - \omega_1\psi_1 \sim \nu^{1/3}$ or less $\to 0$.

Hence: We have found $p + u^2$ continuous, & all depends on $\omega\psi$.

application



p continuous, u^2 cont. $\therefore R$ continuous. (Because $\psi = 0$ where ω jumps)!

R = constant outside, and R = const inside, & is continuous.

$\therefore R$ = const everywhere. Take value as zero. $\therefore \Pi = \omega\psi$ inside & out.

$\therefore R = 0$ thruout region.

But pressure at ∞ must equal mean pressure at $y = 0$ [Prove this.*]

Therefore $\overline{\tfrac12(u^2+v^2) - \omega\psi}\Big|_{y=\infty} = \overline{\tfrac12(u^2+v^2) - \omega\psi}\Big|_{y=0}$

~~Take x means~~ at ∞, $v^2 = 0$, $\omega = 0$ suppose this is possible?

~~Take x means~~ at 0, $v^2 = 0$, $\psi = 0$.

Hence $\overline{u^2}\Big|_\infty = \overline{u^2}\Big|_0$

$\overline{v}_y = -\overline{u}_x = 0$

$\therefore \overline{v}$ = const $= 0$

$^* \int \overline{p_y}\, dy = \int dy \left(\nu\nabla^2 \overset{0}{\overline{v_y}} + \overset{-u_x v = vv_y}{\overline{u v_x}} + \overline{v v_y}\right)$

~~$\int \nabla p = (q \cdot \nabla) q$~~ $\therefore p\Big|_0^\infty = \overline{v^2}\Big|^\infty - \overline{v^2}^{\,0} = 0$ in exact case.

Relation between mean boundry velocity, and mean boundry velocity squared!

actual velocities are $\overset{1+u}{V}$

$$\nu u_{yy} = u u_x + V u_y + u_x \quad (1)$$

$$u_x = -V_y \quad (2)$$

Mean on x:

$$\nu \bar{u}_{yy} = \frac{\partial}{\partial y}(\overline{Vu})$$

$\boxed{(Vu)_\infty = 0}$ assume

$$\nu \bar{u}_y = \overline{Vu}$$

$\therefore$ Incidently, for our case we want to have $\bar{u}_y = 0$, No net drag.

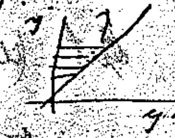
$$\nu u_{yy} = \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(Vu) + u_x \quad (3)$$

$$\nu \bar{u}_0 = -\int_0^\infty \int_0^{2\pi} V(x,y)\, u(x,y)\, dx\, dy = \nu \bar{f} \text{ say.} \quad u(x,0) = f(x)$$

$\therefore$ Desire $\nu \bar{f} = -\int_0^\infty \int_0^{2\pi} V(x,y)\, u(x,y)\, dx\, dy$. Now $V(x,y) = \int_0^y V_y(x,y')\, dy' = -\int_0^y u_x(x,y')\, dy'$

$$\therefore \nu \bar{f} = +\int_0^\infty dy \int_0^{2\pi} dx \int_0^y u_x(x,y')\, u(x,y)\, dy'$$

Integrate with respect to x by parts: $-\int_0^\infty dy \oint dx \int_0^y u(x,y')\, u_x(x,y)\, dy'$



Reverse order, y, y': $-\int_0^\infty dy' \oint dx \int_{y'}^\infty u(x,y')\, u_x(x,y)\, dy$.

Call y, y' & vice versa $= -\int_0^\infty dy \oint dx \int_y^\infty u(x,y)\, u_x(x,y')\, dy'$.

Now consider $\int_y^\infty u_x(x,y')\, dy'$ from (3). $= \int_y^\infty \left(\nu u_{yy} - \frac{\partial}{\partial x}(u^2) - \frac{\partial}{\partial y}(Vu)\right) dy' = -\nu u_y + Vu - \int_y^\infty \frac{\partial}{\partial x} u^2(x,y')\, dy'$.

$$\therefore \nu \bar{f} = \int_0^\infty \oint dx\, dy \left[\nu u_y u\right] - \int_0^\infty \oint V u^2\, dx\, dy + I.$$

IRRELEVENT.
$\Pi = p + \frac{1}{2}u^2 + \frac{1}{2}V^2$
$\omega = u_y - V_x$.
Exact Equations:
$\nu \omega_y = V\omega + \Pi_x$
$\nu \omega_x = u\omega - \Pi_y$

$I = -\int_0^\infty dy \oint dx \left(\int_y^\infty \frac{\partial}{\partial x}(u^2(x,y'))\, dy'\right) u(x,y)$. Integrate parts resp x

$= \int_0^\infty dy \oint dx \left(\int_y^\infty u^2(x,y')\, dy'\right) u_x(x,y) = -\int_0^\infty dy \oint dx \int_y^\infty u^2(x,y')\, dy'\, V_y(x,y)\, dy$ By (2)

Reverse order, y, y' $= -\int_0^\infty dy' \oint dx\, u^2(x,y') \int_0^{y'} V_y(x,y)\, dy = -\int_0^\infty \oint dy'\, u^2(x,y')\, V(x,y')$

$\therefore$ In $\nu \bar{f}$, I cancels the term just preceeding it, so that

$$\nu \bar{f} = \int_0^\infty \oint dx\, dy\, \nu u_y u = \frac{\nu}{2}\overline{u^2(x,0)} = -\frac{\nu}{2}\overline{f^2}.$$

$\therefore$ The mean flow rate adjusts to the mean square velocity!

$\bar{f} = -\frac{1}{2}\overline{f^2}$ ~~But the true~~. Hence $\overline{(1+f)^2} = 1 \quad \therefore \overline{(1+u)^2} = 1$.

Proof two.

To prove: ~~Suppose~~ $\nu[\overline{u(y)} + \frac{1}{2}\overline{u^2(y)}]L] = -\oint dx\, V(x,y)\int_y^\infty [u^2(x,y') + u(x,y')]\,dy'$ ①

It is clearly true as $y \to \infty$ ∴ We prove it by proving its y derivative:

$$\frac{d}{dy}\left(\nu[\overline{u_{(y)}} + \tfrac{1}{2}\overline{u^2(y)}]\right) = +\oint dx\, \underset{\int x \text{ by parts.}}{\overset{=-V_y}{U_x(x,y)}}\int_y^\infty [u^2+u]\,dy' + \overline{Vu^2} + \overline{Vu}$$

$$= -\int dx\, U(x,y)\int_y^\infty [u^2+u]_x\,dy' + \overline{Vu^2} + \overline{Vu}$$

But $(u+u^2)_x + (Vu)_y = \nu u_{yy}$

so integration gives $\int_y^\infty (u+u^2)_x\,dy' = -\nu u_y + Vu$ ②

$$\therefore \underline{RHS} = -\int dx\,[-\nu u u_y + Vu^2] + \overline{Vu^2} + \overline{Vu}$$

$$= +\nu\overline{u_y u} + \overline{Vu}$$

while the mean of ① tells us that $0 = -\nu\overline{u_y} + \overline{Vu}$ ③

$$\therefore RHS = \nu(\overline{u_y u} + \overline{u_y}) = \frac{d}{dy}\nu[\overline{\tfrac{u^2}{2}} + \overline{u}] \quad \underline{QED}$$

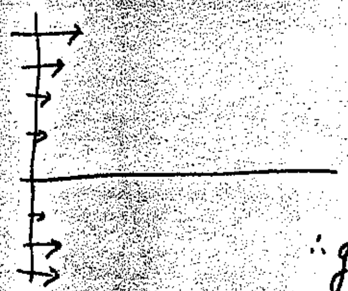
Special consequence of ①:

for $y=0$, $V=0$ hence $\nu(\overline{u(0)} + \frac{1}{2}\overline{u^2(0)}) = 0$ i.e. $\overline{(1+u)^2} = 1$

also ③ says $\overline{u_y(0)} = 0$

Hence, in equilibrium the root mean square surface velocity is the velocity which determines the effective flow, such that the average drag is zero.

Boundary Shear Layer



$u_y\big|_{x=0} = 1$ $\qquad \psi = x^{2/3}\, g\left(\frac{y}{x^{1/3}}\right)$ $\qquad u = \psi_y$, $v = -\psi_x$

$u_y\big|_{y=0} = 0$

$\nu u_{yy} = u u_x + v u_y$

$u_y = g''(z)$ $\qquad 3\nu g'''(z) = g'g' - 2gg''$ $\qquad z = \frac{y}{x^{1/3}}$

$\therefore g''(0) = 0$

$g''(\infty) = 0$ $\qquad$ Try $g(0) = 0$. (Stream function $= 0$ at center)

(Note if put $z' = \nu z$, then ν goes out.
Therefore $\frac{y\nu}{x^{1/3}}$ is better variable)

$\frac{3\nu}{2} \cancel{3} g''' = \cancel{2}\frac{1}{2} g'g' - \cancel{2} gg''$.

Now if f is another solution, satisfying $f(0) = 0$, $f'(0) = 1$, $f''(0) = 0$, $f''(\infty) = a$.

Then note that $g = b\,f(bz)$ is a solution $\quad g(0) = 0$, $g'(0) = b^2$, $g''(0) = 0$, $g''(\infty) = ab^3$.

$\therefore$ We choose $b^3 a = 1$ later.

$\frac{3\nu}{2} f''' = \frac{1}{2}(f')^2 - ff''$. $\qquad f''' = \frac{1}{2}\cdot\left(\frac{2}{3\nu}\right) = \frac{1}{3\nu}$ $\quad f'''' = 0$ $\quad f''''' = -\frac{1}{2}\left(\frac{2}{3\nu}\right)^3$

$\frac{3\nu}{2} f'''' = -ff'''$ $\qquad f = z + \frac{1}{12} z^3 - \frac{1}{2\cdot 5!} z^5 \ldots$

$\therefore f'''(z) = \frac{1}{3\nu}\exp\left(-\int_0^z f(x)\,dx \cdot \frac{2}{3\nu}\right)$ $\qquad f''' = \frac{1}{2} - \frac{1}{4}z + \ldots$

Eg. 1st approx. $\quad$ Zero Approx $= z$.

$f_1'''(z) = \frac{1}{3\nu} e^{-z^2/3\nu}$ $\qquad \cancel{f_1(z) = -\frac{1}{2}e^{-z}}$ $\qquad \frac{1}{2} - \frac{1}{4}z^2 + \frac{z^4}{4\cdot 2}$

$f''(\infty) = \frac{1}{3\nu}\int_0^\infty e^{-z^2/3\nu}\,dz = \frac{\sqrt{\pi}}{2\sqrt{3\nu}} = a.$ $\qquad \cancel{f = z + \frac{1}{2}}$ $\quad b = \frac{1}{a^{1/3}} = \left(\frac{2\sqrt{3\nu}}{\sqrt{\pi}}\right)^{1/3}$ Nearly.

$\therefore \boxed{g_1'''(z) \approx \frac{b^4}{3\nu} e^{-\frac{z^2 b^2}{3\nu}}}$ $\qquad b = \nu^{1/6}\left(\frac{2\sqrt{3}}{\sqrt{\pi}}\right)^{1/3} \approx 1.25\,\nu^{1/6}$ $\qquad \frac{b^4}{3\nu} = \frac{2\sqrt{3}}{\sqrt{\pi}}\cdot\frac{1.25\,\nu^{1/6}}{3\nu}$

GOOD APPROX.

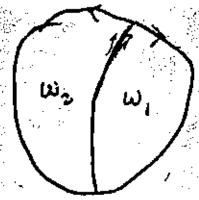
$g'(0) = b^2 = 1.57\,\nu^{1/3}$

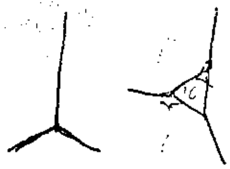
$\alpha = \left(\frac{3\sqrt{\pi}}{2}\right)^{1/3} = 1.37$

$\mathrm{Erf} = \frac{2}{\sqrt{\pi}}\int_0^y e^{-x^2}\,dy$ $\qquad u_y = \ldots$ $\qquad \mathrm{Erf}\left[\frac{y}{(\alpha\nu x)^{1/3}}\right] = u_y$

NEARLY EXACT. $\qquad$ ANS.

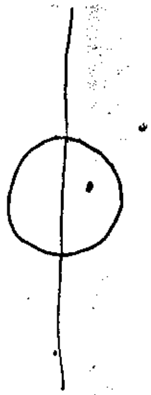
$\rightarrow$ In case initial shear is ω, instead of 1, replace ν by $\frac{\nu}{\omega}$





$$\frac{\partial \omega}{\partial \psi}\left(\int \omega\, d\mathrm{Vol}\right) = 0$$

$$\nabla^2 \psi = \omega$$



$$\psi(x,y) = \int \ln\left(\frac{r_1^2}{r_2^2}\cdot\frac{x'^2+y'^2}{a^2}\right)\omega(x',y')\,dx'dy'$$

$$r_1^2 = x^2+y^2 - 2(xx'+yy') + (x'^2+y'^2)$$

$$r_2^2 = x^2+y^2 - \frac{2a^2(xx'+yy')}{x'^2+y'^2} + \frac{a^4}{x'^2+y'^2}$$

If x,y is on circle, then $r_2^2 = \frac{a^2}{x'^2+y'^2}\, r_1^2$

$$V_x = \int 2\omega(x',y')\left[\frac{y-y'}{r_1^2} - \frac{y - \frac{a^2 y'}{x'^2+y'^2}}{r_2^2}\right]dx'dy'$$

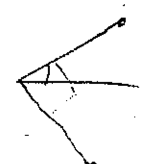
On circle, V is tangential. V_{tang} on circle $= q_t = 2a\int \omega(x',y')\left[1-\frac{x'^2+y'^2}{a^2}\right]\frac{dx'dy'}{a^2-2(xx'+yy')+x'^2+y'^2}$

$$q_t(\beta) = 2a\int_0^a\int_0^{2\pi}\omega(\rho,\alpha)\left(1-\frac{\rho^2}{a^2}\right)\frac{\rho\,d\rho\,d\alpha}{a^2-2a\rho\cos(\alpha-\beta)+\rho^2}$$

Example $\omega = +\omega_0$ if $\alpha = 0$ to π, $= -\omega_0$ for $\alpha = \pi$ to 2π, suppose β is in 0 to π

$\alpha = \beta + \gamma \quad \therefore \omega = +\omega_0$ for $\gamma = \beta$ to $\pi+\beta$ $\quad \cos(\gamma-\pi) = -\cos\gamma$

$-\omega_0$ for $\gamma = \beta - \pi$ to β



$\sqrt{a^2-b^2} = a^2+\rho^2$

$$\therefore \int_{\beta-\pi}^{\beta} \; a^2 \mp \; = 2a\omega_0\int 2\left(1-\frac{\rho^2}{a^2}\right)\frac{\rho\,d\rho}{a^2-\rho^2}\,\pi\,\tan^{-1}\left(\frac{\sqrt{\frac{a+\rho}{a-\rho}}\,\sqrt{a^2-\rho^2}\,\tan\frac{1}{2}\beta}{(a-\rho)^2}\right) + \tan^{-1}\left(\sqrt{\frac{a+\rho}{a-\rho}}\,\tan\left(\frac{\beta}{2}\right)\right)$$

$$+\tan^{-1}\left(\sqrt{\frac{a-\rho}{a+\rho}}\right) - \tan^{-1}\left(\sqrt{\frac{a-\rho}{a+\rho}}\right)$$

$$= 4a\omega_0\int\frac{2\rho\,d\rho}{a^2}\left\{\tan^{-1}\left(\sqrt{\frac{a+\rho}{a-\rho}}\tan\tfrac{1}{2}\beta\right) - \tan^{-1}\left(\sqrt{\frac{a-\rho}{a+\rho}}\tan\tfrac{1}{2}\beta\right)\right\}$$

$$\therefore q_t(\beta) = 8a\omega_0\int_{-1}^{+1} u\,du\,\tan^{-1}\left(\frac{1+u}{1-u}\tan\tfrac{1}{2}\beta\right).$$

$$\nu \omega_y = V\omega - \Pi_x$$
$$\nu \omega_x = U\omega + \Pi_y$$

Let $\tilde{\omega} = \text{const}$

$\tilde{\Pi}$, $\tilde{u}$, $\tilde{v}$ = values of u, v going with it

$$\tilde{\Pi}_x = \tilde{V}\tilde{\omega}$$
$$\tilde{\Pi}_y = -\tilde{u}\tilde{\omega}$$

$\tilde{q}_n = 0$

$$q = \tilde{q} + q'$$
$$\omega = \tilde{\omega} + \omega' \text{ etc}$$

$\int q_t' ds = 0$

$\cancel{\nu\omega' = \tilde{V}\omega}$

$$-\nu \nabla\times\omega = -\tilde{V}\times\omega - V\times\tilde{\omega} - V\times\omega - \Pi_x$$

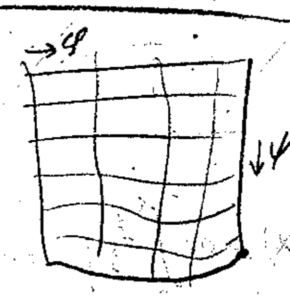
$$\int \tilde{q}_t \omega ds + \underbrace{\int \omega dv_n}_{\tilde{\omega}} = \int (\tilde{q}\circ(\nabla\times\omega)) dVol = -\int(\tilde{q}\times\tilde{\omega})\circ q \, dVol - \int \tilde{q}\circ(q\times\omega) \cancel{- \int \tilde{q}\circ\nabla\Pi}$$

$\tilde{\omega}\int q_t ds$

$\overset{\shortparallel}{0}$ supposed

$$u = U + u \qquad v = V + v$$

$$U = \frac{\partial\varphi}{\partial x} = +\frac{\partial\psi}{\partial y}$$
$$V = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$$



for $\varphi \sim U_0 x$

$\psi \sim U_0 y$

Bottom $\psi = \text{const} = 0$.

$$\nu\omega_y = V\omega - \Pi_x \qquad \frac{\partial\omega}{\partial x} = \frac{\partial\omega}{\partial\varphi}U - V\frac{\partial\omega}{\partial\psi}$$
$$\nu\omega_x = U\omega + \Pi_y \qquad \frac{\partial\omega}{\partial y} = \frac{\partial\omega}{\partial\varphi}V + U\frac{\partial\omega}{\partial\psi}$$

$$\lambda = \frac{x e}{\hbar e} \qquad n = \frac{\hbar e}{\hbar e}$$

$$\int pV\,dy = \int p\int_y^\infty u_x dy'\,dy = \int p_x \int_0^\infty u\,dy'\,dx$$

$$\nu(U_{xx} + U_{yy}) - (UU_x + VU_y)$$

$\omega \sim$

$\sim$

m

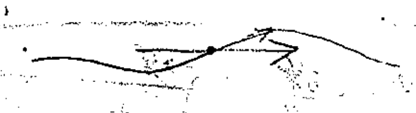
$m + m$

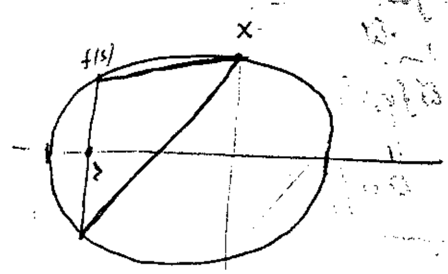
$$f(x) = \omega\int_{-a}^{a}\left\{\left[-2f\ln\left(\frac{(x-\xi)^2+f(\xi)^2}{(x-\xi)^2+4f^2}\right) - 2(x-\xi)\left[2\tan^{-1}\frac{f}{x-\xi} - \tan^{-1}\frac{2f}{x-\xi}\right]\right\}d\xi$$

$$f'(x) = \omega\int_{-a}^{a}\left\{\left[-\frac{4f(x-\xi)}{(x-\xi)^2+f^2} + \frac{4f(x-\xi)}{(x-\xi)+4f^2} - 2\left[2\tan^{-1}\frac{f}{x-\xi} - \tan^{-1}\frac{2f}{x-\xi}\right]\right\}d\xi$$

$$-2(x-\xi)\left[-2\frac{f}{(x-\xi)^2+f^2} + \frac{2f}{(x-\xi)^2+4f^2}\right]$$

$$f'(x) = 2\omega\int_{-a}^{a}\left\{2\tan^{-1}\left[\frac{f(\xi)}{x-\xi}\right] - \tan^{-1}\left[\frac{2f}{x-\xi}\right]\right\}d\xi$$





$$f(x) = \omega\int_{-a}^{a}d\xi\int_0^{f(\xi)}d\eta\,\ln\left(\frac{(x-\xi)^2+(f(x)-\eta)^2}{(x-\xi)^2+(f(x)+\eta)^2}\right)$$

$$f'(x) = 2\omega\int d\xi\int_0^{f(\xi)}d\eta\cdot\frac{(x-\xi)+(f(x)-\eta)f'(x)}{(x-\xi)^2+(f(x)-\eta)^2} - \quad +$$

$$+$$

$$\int_{-a}^{a}d\xi\left[-2\tan^{-1}\left(\frac{f(x)-f(\xi)}{x-\xi}\right) + 4\tan^{-1}\frac{f(x)}{x-\xi} - 2\tan^{-1}\frac{f(x)+f(\xi)}{x-\xi}\right.$$

$$+f'(x)\ln\left(\frac{[(x-\xi)^2+f(x)^2]^2}{[(x-\xi)^2+(f(x)-f(\xi))^2][(x-\xi)^2+(f(x)+f(\xi))^2]}\right)$$

$\rho = a\cos\theta$

$$8aw_0\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\left\{\tan^{-1}\left(\frac{\tan\frac{1}{2}\beta}{\tan^2\frac{1}{2}\theta}\right)-\tan^{-1}\left(\tan\tfrac{1}{2}\beta\tan^2\tfrac{1}{2}\theta\right)\right\}$$

$$\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\,\tan^{-1}\frac{\tan\frac{1}{2}\beta}{\tan\frac{1}{2}\theta}=\int_{\pi/2}^{\pi}-\cos\theta\sin\theta\,d\theta\,\tan^{-1}\left(\frac{\tan\frac{1}{2}\beta}{\tan\frac{1}{2}\theta}\right)$$

$$8aw_0\int_0^{\pi/2}\cos\theta\sin\theta\,d\theta\,\tan^{-1}\left(\frac{\tan\frac{1}{2}\beta}{\tan^2\frac{1}{2}\theta}\right)$$

$$\tan^{2}\tfrac{1}{2}\theta=\left(\tan\tfrac{1}{2}\beta\right)\tan X=\frac{\sin\theta}{1+\cos\theta}$$

$$\frac{d\theta}{1+\cos\theta}=\left(\tan\tfrac{1}{2}\beta\right)\frac{dX}{\cos^2X}$$

$$\cos\theta\sin\theta\,d\theta=\cos\theta\sin\theta(1+\cos\theta)\left(\tan\tfrac{1}{2}\beta\right)\frac{dx}{\cos^2X}$$

$$\int_{-a}^{a}\rho\,d\rho\,\tan^{-1}\left(\frac{a+\rho}{a-\rho}\tan\tfrac{1}{2}\beta\right)$$

$$\frac{a-\rho}{a+\rho}=X\tan\tfrac{1}{2}\beta$$

$$\rho=a\frac{1-xt}{1+xt}\qquad t=\tan\tfrac{1}{2}\beta$$

x goes 0 to ∞

$$d\rho=a\frac{1}{(1+xt)^2}$$

$$a^2t\int_0^\infty\frac{1-xt}{(1+xt)^2}dx\,\left(\frac{\pi}{2}-\tan^{-1}X\right)$$

$$a^2t\int_0^{\pi/2}\frac{1-t\tan u}{(1+t\tan u)^2}u\,du$$

$$X=\frac{1}{\tan u}$$

$$\frac{q_t(\beta)}{8aw_0}=\int_{-1}^{+1}$$

$$\tan\tfrac{1}{2}(\pi-\theta)=\frac{1}{\tan\frac{1}{2}\theta}$$

$$\int_{-a}^{0}\rho\,d\rho\,\tan^{-1}\left(\frac{a+\rho}{a-\rho}x\right)=\int_0^a\rho\,d\rho\,\tan^{-1}\frac{a-\rho}{a+\rho}x$$

$$\left[1-\left(\frac{X-D}{X+D}\right)\ln\left(\frac{2X}{2D}+1\right)\frac{\pi}{X}\right]\left(\frac{2X}{2D}-1\right)=$$

$$\int_{-1}^{+1} u\,du\,\tan^{-1}\left(\frac{1+u}{1-u}t\right) = \frac{u^2}{2}\tan^{-1}\left(\frac{1+u}{1-u}t\right)\Big|_{-1}^{+1} - \int_{-1}^{+1} u^2 \frac{\frac{t}{(1-u)^2}\,du}{1+\left(\frac{1+u}{1-u}t\right)^2}$$

$\frac{1+u}{1-u} = 1 + \frac{2u}{1-u}$

$$= \frac{\pi}{4} - t\int_{-1}^{+1} \frac{u^2\,du}{(1+t^2)(1+u^2)+2u(t^2-1)} = \frac{\pi}{2} - \frac{t}{1+t^2}\int_{-1}^{+1}\frac{u^2\,du}{1+u^2+2ux}$$

$$X = \frac{t^2-1}{t^2+1}$$

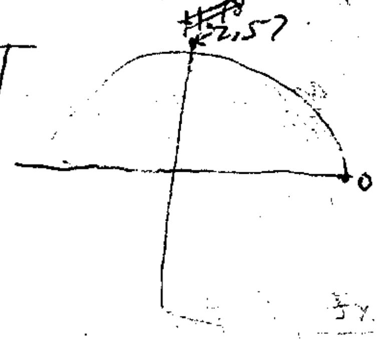
$$= \frac{\pi}{4} - \frac{t}{1+t^2}\left[u - x\ln(1+u^2+2ux) + \frac{4x^2-2}{2}\frac{2}{\sqrt{4(1-x^2)}}\tan^{-1}\left(\frac{2u+2x}{2\sqrt{1-x^2}}\right)\right]_{-1}^{+1}$$

$$= \frac{\pi}{4} - \frac{t}{1+t^2}\left[2 - x\ln\left(\frac{1+x}{1-x}\right) + \frac{2x^2-1}{\sqrt{1-x^2}}\left[\tan^{-1}\sqrt{\frac{1+x}{1-x}} + \tan^{-1}\sqrt{\frac{1-x}{x+1}}\right]\right]$$

$t = \tan\frac{1}{2}\beta$, $\quad x = \dfrac{\frac{1-c}{1+c}-1}{\frac{1-c}{1+c}+1} = \dfrac{-2c}{2} = -\cos\beta \quad \Big| \quad \dfrac{t}{1+t^2} = \dfrac{\sin\beta}{1+\cos\beta}\dfrac{1}{\left(1+\frac{1-\cos}{1+\cos}\right)} = \frac{1}{2}\sin\beta$

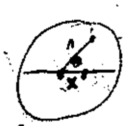
$$\frac{\pi}{4} - \frac{1}{2}\sin\beta\left[2 + 2\cos\beta\ln\tan\tfrac{1}{2}\beta + \frac{2\cos^2\beta-1}{\sin\beta}\left[\frac{\pi}{2}\right]\right]$$

$$\boxed{\frac{q_t}{b a \omega_0} = -\frac{\pi}{4}\sin^2\beta - \sin\beta\left(1+\cos\beta\ln\tan\tfrac{1}{2}\beta\right)}$$



$$= -\sin\beta\left\{1+\cos\beta\ln\tan\tfrac{1}{2}\beta + \frac{\pi}{4}\sin\beta\right\}$$

along Diameter $\quad y=0 \quad V_x = \int 2\omega(x',y')\left[\dfrac{-y'}{x^2-2xx'+x'^2+y'^2} - \dfrac{-\frac{a^2y'}{x'^2+y'^2}}{x^2-\frac{2a^2x}{x'^2+y'^2}x'+\frac{a^4}{x'^2+y'^2}}\right]dx'\,dy'$



$$= \int_0^a\int_0^\pi 4\omega_0\rho^2\,d\rho\,d\alpha\left[\frac{-1}{x^2-2x\rho\cos\alpha+\rho^2} + \frac{a^2}{\rho^2x^2-2a^2x\rho\cos\alpha+a^4}\right]\sin\alpha.$$

$$= \int_0^a 4\omega_0 a\rho^2\,d\rho\left[\frac{1}{x\rho}\ln\left(\frac{x+\rho}{x-\rho}\right) - \frac{1}{x\rho}\ln\left(\frac{\rho x+a^2}{\rho x-a^2}\right)\right] = \frac{4\omega_0 a}{x}\left[\cancel{(x+a)(\ln(x+a)-1)}\right.$$

$$= 4\omega_0 a\left(1-\frac{a^2}{x^2}\right)\left[\frac{x}{2a}\left(1+\frac{a^2}{x^2}\right)\ln\frac{a+x}{a-x} - 1\right]$$

see over

$\cancel{\left[(x+\rho)\ln(x+\rho) - (\rho-x)\ln(\rho-x) - 2x - \frac{\rho x+a^2}{x}\ln(\rho x+a^2) + \frac{\rho x-a^2}{x}\ln(\rho x-a^2) - \frac{2a^2}{x}\right]_0^a} \; \cancel{-\ln\frac{a^2}{x^2} + \left(1-\frac{a^2}{x^2}\right)\ln}$

$\int_0^\infty \overline{VU}\,dy$

$V_y U = -U U_x$

$V U_y = V$

$\rho \nabla^2 \mathbb{V} = (\mathbb{V}\cdot\nabla)\mathbb{V} + \nabla p$

$= \mathbb{V}\times\omega + \nabla \Pi.$

$V = \int_y^\infty \nu U_{yy} - U U_x -$

$U U_x + V U_y = \left(\frac{U^2}{2}\right)_x$

$V(U_y - V_x) \quad + W U_z \quad - V V_x$

$W(U_z - W_x) \quad - W W_x$

$V\omega_z^2 - W\omega_y^2$

$(UV)_y = V U_y + U V_y = V U_y + U(-\nu U_{yy} + U U_x + V U_y)$

$\nu \nabla^2 U = V U_y + \Pi_x$

$-V V_x$

$\nu \nabla^2 V = U V_x - U U_y + \Pi_y$

$\overline{(UV)}_y = \overline{V U_y} - \nu \overline{U_{yy} U} + \overline{U V U_y}$

$\boxed{\omega = U_y - V_x}$

$U_x = -V_y$

$\overline{V(U^2_{\ })_y} = \frac{\partial}{\partial y}\left(\overline{V U^2/2}\right)$

$\nu \nabla^2 U = V\omega + \Pi_x = \nu\omega_y$

$-V_{xy} + V_{xx}$

$-\nu \nabla^2 V = U\omega + \Pi_y = -\nu\omega_x$

$-V_{yx} + U_{yy}$

$= \omega_y$

$\overline{(UV)}_y = -\nu \overline{U U_y} - \int_y \nu \overline{U_y^2}\,dy$

$\nu(U\omega_y - V\omega_x) = U\Pi_x + V\Pi_y$

$= (U\Pi)_x + (V\Pi)_y$

$\int U \int_y U_x = \int U \int_y [\nu U_{yy} + (VU)_y + \partial_x(U^2)]$

$= \int V_y \int_y U^2 + U V U$

$(\mathbb{V}\cdot\nabla) U$

$-V\int_y U^2 - V U^2$

$\therefore$ if say $\int_0^\infty U(x,y) \int_y^\infty \left[(VU)_y + (UU)_x\right]_{y'} dy'\, dx\, dy = 0$

$(U^2+U)_x = \nu U_{yy} - (VU)_y$

$\frac{d}{dy}\, \nu[U + \tfrac{1}{2}U^2] = + \int dx\, U_x \int_a^\infty [U^2 + U]\, dy' + V(U^2 + U)$

$\nu[(\mathbb{V}\times\nabla)\cdot\omega] = \nabla\cdot(U\Pi)$

$= + \int dx\, U \cdot \nu U_{yy} - \int dx (VU^2)$

$\Pi\Big|_0^\infty = \iint \overline{U\omega}\, dx\, dy$

$\cancel{= \iint U(\nabla\times\mathbb{V})\, dx\, dy}$

$\int \overline{U\omega} = \int U \int_y^\infty (\Pi_x + (V\omega))\, dy'\, dx\, dy$

$U \int_y^\infty V(U_y - V_x)\, dy'\, dx\, dy$

$U \int_y \Pi_x = + V_y \int_y \Pi = \int (V\Pi)\, dx$

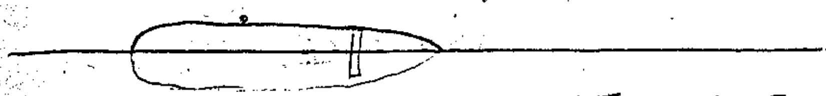
$V U^2$

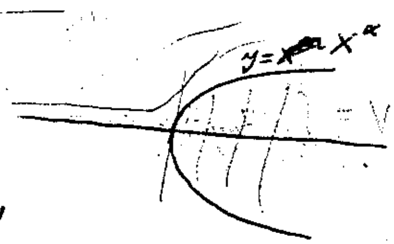
$\int (U U_y - U V_x) = \frac{U^2}{2}\Big|_0^\infty -$

$V_y \int (U_y U - V V_x)$

$\int_y^\infty (2u+1)_y V\, dy$

$\overline{VU_y} \quad \overline{V_x U}$





$\cancel{u\,U} \quad \cancel{u\,U}$

$(u-U)^2 + (u-U)U$

$\psi_{yy} = \omega(x,y)$

$= u^2 + u - uU$

$V_y = U_x$

$\omega = u_y - V_x \propto U_y$

$U = \psi_y = \int_0^y \omega(x,y')\,dy' = \omega f(x) + b(x) \text{ for } y > f$

$= \omega y + b(x) \quad \text{for } y < f$

$\frac{\partial}{\partial Y} \sim \frac{1}{\epsilon}$

$V \sim \epsilon$

$\psi_x \sim \epsilon$

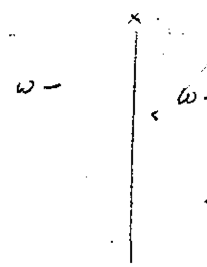
$\psi_y \sim 1 \quad \psi \sim \epsilon$

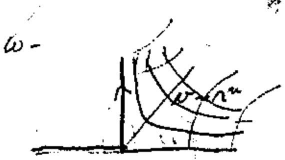
$\psi_y = 1 \quad y > f$

$= 1 - \omega(f(x) - y) \quad y < f(x).$

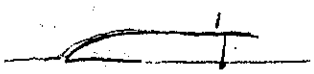
$\overline{V(u+v)_x} = \overline{V u_x} - \ldots$

$f(x) = x^\alpha \qquad \psi_{yy} + \psi_{xx} = \omega$



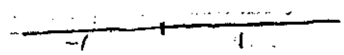


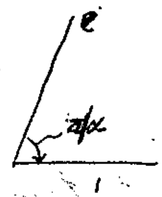
$1 - \omega f(x) = G(x) < 0$



$f(x) > \frac{1}{\omega}$

call ω very large





R.P. z^α

$\varphi = (x+iy)^\alpha + (x-iy)^\alpha$

$\psi = (x+iy)^\alpha + (x-iy)^\alpha$

Example.

$f(\xi) = \sqrt{\xi}$

$\int_0^L (\ln(\xi+1) - \ln \xi)\, d\xi \to$ tractable.

$V_x = \alpha(x+iy)^{\alpha-1} - (x-iy)^{\alpha-1}$

$\alpha \rho^{\alpha-1}\left(e^{i\theta(\alpha-1)} - e^{-i\theta(\alpha-1)}\right)$

$\psi(0) = 0 + \int \ln 1 = 0.$

$u_x = \psi_y = 1 + \omega \iint \frac{-2\eta}{\xi^2+\eta^2}\, d\eta\, d\xi$

$$\begin{cases} V_y = \alpha\rho^{\alpha-1} \sin\theta(\alpha-1) \\ V_x = \alpha\rho^{\alpha-1} \cos\theta(\alpha-1) \end{cases}$$

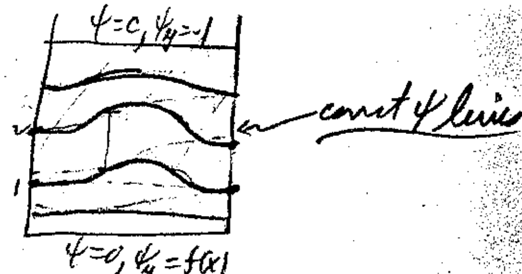
$= 1 + \omega \int_0^\xi \ln\left[\frac{\xi^2 + f(\xi)^2}{\xi^2}\right] d\xi.$

$90^\circ: \quad V \sim \rho$

$\frac{\partial V_x}{\partial y} - V_{yy} = u_{xy} \ldots = \omega \iint \frac{2\, d\eta}{\xi^2+\eta^2}\, d\xi$

$\omega = J^2 \psi_{yy}$; $u = \psi_y, \; v = \psi_x$

$\nu \omega_{yy} = \psi_y \omega_x - \psi_x \omega_y.$



Consider $\oint_{\text{streamline}} J^2 W(x) \left(\frac{\partial q_t}{\partial n}\right) dx = \oint W(x)\, \underline{\omega}\, dx$ $\quad \sim \frac{1}{J}$

$= \oint J \check{W}(x) \frac{\partial q_t}{\partial \psi} \frac{\partial \psi}{\partial n} dx = \oint J^2 \check{W}(x) \frac{\partial q_t}{\partial \psi} q_t\, dx = \frac{1}{2}\frac{d}{d\psi}\left[\oint J^2 \check{W}(x)\, q_t^2\, dx\right]$

$- \frac{1}{2}\int \frac{dWJ^2}{d\psi} q_t^2 dx$ finite

$\therefore \frac{1}{2}\Delta\left[\oint W(x) J^2 q_t^2\, dx\right] = \int d\psi \oint \underline{W(x)}\, \omega\, dx$

Now all we must do is find $W(x)$ so $\oint W(x)\,\omega\, dx$ __along a streamline__ is finite.

$\int d\psi \oint W(x)\omega\, dx = \iint W(x)\,\omega\, dx \cdot \frac{\partial \psi}{\partial y} dy = \int W(x)(\omega \psi_y)\, dx\, dy = \frac{1}{2}\int W(x) J^2 \frac{d}{dy}(\psi_y^2)\, dy = \left|\int W(x) J^2 \psi_y^2\right| dx$

Area $\iint W(x)\, \psi \omega_y\, dx\, dy$

~~consider $\nu W(x)\psi\omega_{yy} = W(x)\psi\psi_y\omega_x - W(x)\psi\psi_x\omega_y$~~

But we know that $\nu\omega_y = JJ_x\psi_y^2 + J^2\psi_{xy}\psi_y - J^2\psi_x\psi_{yy} - JJ_x$

$\nu W\psi\omega_y = WJJ_x\psi_y^2\psi + WJ^2\psi_{xy}\psi_y\psi - WJ^2\psi_x\psi_{yy}\psi - WJJ_x\psi$

$= -\frac{\partial}{\partial y}(WJ^2\psi_x\psi_y\psi) + \frac{\partial}{\partial x}(WJ^2\psi_y^2\psi) - (WJJ_x + W_xJ^2)\psi_y^2\psi - WJJ_x\psi$

Integrate over area beneath a streamline

$\nu \int_0^{\text{str.}} W\psi\omega_y\, d\text{area} = \underbrace{\cancel{\int(-N_y\psi_x + N_x\psi_y)}}_{=0 \text{ for streamline}} WJ^2\psi\psi_y - \int_0^{\text{str.}} [(WJJ_x + W_xJ^2)(\psi_y^2\psi) + WJJ_x\psi]\, dy\, dx$

$\int \psi_v^2\, d\mu = \int$

$\nu W\psi\psi_{yyy} = \nu WJ^2\frac{d}{dy}\left(\psi\psi_{yy} - \frac{1}{2}\psi_y^2\right)$

$\omega = J^2\psi_{yy} \qquad \omega_y = J^2\psi_{yyy} \quad \omega_x = J^2\psi_{xyy} + 2JJ_x\psi_{yy}$

$\tilde{J}J_\mu = \bar{u}\bar{u}_x + \bar{v}\bar{v}_y = \bar{p}_x$

$\nu\omega_{yy} = \psi_y\omega_x - \psi_x\omega_y$

$\nu J^2\psi_{yyyy} = J^2\psi_y\psi_{xyy} + 2JJ_x\psi_y\psi_{yy} - J^2\psi_x\psi_{yyy}$

$d\mu\, d\nu =$

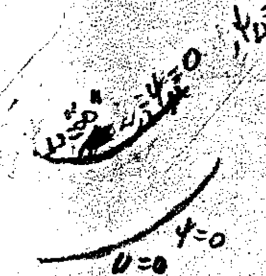
$= \frac{\partial}{\partial y}\left(JJ_x\psi_y^2 + J^2\psi_{xy}\psi_y - J^2\psi_x\psi_{yy}\right)$

$\therefore \nu J\psi_{yyy} = J_x\psi_y^2 + J\psi_{xy}\psi_y - J\psi_x\psi_{yy} + \ldots J_x \qquad (\psi_{yyy} = 0 \text{ at } \infty$, $\psi_x = 0$

$\nu J\psi_{yyy} = \frac{\partial}{\partial x}(J\psi_y^2) - \frac{\partial}{\partial y}(J\psi_y\psi_x) \quad + \ldots J_x$ at ∞: ~~$\psi_y = u_\infty = \bar{u} = J$~~, $J^2\psi_x = 0$, $J^2\psi_y = -1$

Multiply ψ:

$\nu J\psi\psi_{yyy} = \frac{\partial}{\partial x}(\psi J\psi_y^2) - \frac{\partial}{\partial y}(J\psi_y\psi_x\psi) - J_x\psi$



Integrate x: $\nu\frac{d}{dy}\left(\overline{J\psi\psi_{yy}} - \overline{\tfrac{1}{2}J\psi_y^2}\right) = \frac{d}{dy}\overline{J\psi\psi_y\psi_x} + \overline{J\psi_x}$

$\therefore \overline{J\psi\psi_{yy}} - \overline{\tfrac{1}{2}J\psi_y^2} - \overline{J\psi\psi_y\psi_x} = \text{const} + \int^y \overline{J\psi_x}\,dy$

Taken between upper & lower limits,

$\overline{J\psi_\infty\omega_\infty} - \tfrac{1}{2}\overline{J\psi_{y\infty}^2} + \tfrac{1}{2}\overline{J\psi_y^2}\Big|_0 = \int_0^{\text{“}\infty\text{”}}\!\!\int J\psi_x\,dx\,dy$

$\int J\psi_y^2\,dx = \int J\psi_\nu^2\,d\mu = \int \frac{J}{J\psi}(\bar{v}\psi_\xi - \bar{u}\psi_\eta)^2\,d\mu = \int\frac{1}{J^3}(\bar{v}v + \bar{u}u)^2\,d\mu$ old streamline (const ν)

Now $\bar{q}_t\,\bar{V} = \nabla\mu \quad \therefore \int_{\text{old stream}}\frac{1}{J^3}(\bar{v}v + \bar{u}u)^2\bar{q}_t\,ds = \int q_t^2\,ds$ if old & new coincide (then $\bar{v}v + \bar{u}u = q_0\bar{q}_t$, $J = \bar{q}_t$)

$\int J\psi_x\,dx\,dy = \int J\psi_\mu\,d\mu\,d\nu = \int J(\bar{u}\psi_\xi + \bar{v}\psi_\eta)\,d\xi\,d\eta = \int\bar{q}(\bar{q}\times q)\,d\text{area} \qquad d\mu\,d\nu = J^2\,d\xi\,d\eta$

$= \int\bar{q}_t^2(q_t + \ldots)\,d\text{area}$?!?!

$\int J_\mu\psi\,d\mu\,d\nu = \int J^2J_\mu\psi\,dx\,dy = \int\bar{p}_x\psi\,d\text{area}$

$= \int p\,\bar{q}_m\,d\text{area}$

$$\overline{J\psi\psi_{yy}\psi_x} + \overline{J\psi\psi_y\psi_{yx}}$$

$$\Phi 2\psi_y \psi_{yx} - \psi_{yy}\psi_x - \psi_y \psi_{xy}$$

$$\frac{\partial}{\partial y}\overline{\tfrac{1}{2}(J\psi^2\psi_{xy})} = J\psi\psi_y\psi_{xy} + \frac{J}{2}\psi_x\psi_{yy}\psi - J_x\frac{\psi^2}{2}\psi_{yy}$$

EXACT VISCOUS EQUATIONS

$\nabla \cdot \mathbb{V} = 0$

$\nu \nabla^2 \mathbb{V} = (\mathbb{V} \cdot \nabla)\mathbb{V} + \nabla p$

$\boxed{\Pi = p + \frac{1}{2}(u^2+v^2+w^2)}$

Or $\; +\nu \nabla \times \omega = -\mathbb{V} \times \omega + \nabla \Pi$

$(\mathbb{V}\cdot\nabla)\mathbb{V}\Big|_x = uu_x + vu_y + wu_z$

$= uu_x + v(u_y - v_x) + w(u_z - w_x)$
$+vv_x$
$+ww_x$

$= \left(\frac{1}{2}(u^2+v^2+w^2)\right)_x - v\omega_3 + w\omega_2$

$\omega_1 = w_y - v_z$
$\omega_2 = u_z - w_x$
$\omega_3 = v_x - u_y$

Two dimensions:

$$\boxed{\begin{aligned} +\nu\omega_y &= +V\omega - \Pi_x \\ \nu\omega_x &= U\omega + \Pi_y \end{aligned}}$$

$V_y = -U_x$

$\omega = V_x - U_y.$

$\nu(\omega_{xx} + \omega_{yy}) = V\omega_y + U\omega_x$

Case periodic in x boundry, Bar means av. on x: $\bar{A}(y) = \int_0^{2\pi} A(x,y)\,dx$

av. Eqn. ② $\quad \overline{\Pi}_y = -\overline{U\omega} = -\overline{UV_x} + \overline{UU_y} = +\overline{V_yV} + \overline{UU_y} = +\frac{1}{2}\frac{\partial}{\partial y}(u^2 - v^2)$

Integrating $\left(\overline{\Pi} - \frac{1}{2}\overline{u^2} + \frac{1}{2}\overline{v^2}\right) = \text{const} = \bar{p} + \overline{v^2}$

av Eqn ①. $\quad \nu\overline{\omega}_y = \overline{V\omega} = \cancel{\overline{VV_x}} - \overline{VU_y} = -\nu\,\overline{U}_y = -\frac{\partial}{\partial y}(\overline{VU}) + \cancel{\overline{V_yU}}$

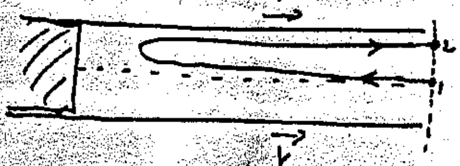
$\therefore \nu\overline{U}_y - \overline{VU} = \text{const}_0 \; \cancel{=\nu\overline{U}_0}$

Integrate eqn. ② respect to y, 0 to ∞: $V_0 = 0,\ V_\infty = 0,\ U_\infty = U = \text{const}\quad p_\infty = 0.\quad \omega_\infty = 0$

$\nu\int_0^\infty (V_{xx} - U_{xy})\,dy = \int_0^\infty (UV_x)\,dy = \int_0^\infty (UV_x + U_xV)\,dy = \frac{\partial}{\partial x}\int (UV)\,dy \qquad \nu\overline{U}_y = \overline{VU}$

Int. x: $\quad \nu\int_0^\infty (V_x - U_y)\,dy - \int (U\omega)\,dy = \text{CONST.}$

Open Long Wake



Suppose wake is as shown, roughly.

Then go down far enough that velocities are nearly $\parallel$ ~~to plane~~

Then we know:

$$\Pi_2 - \Pi_1 = \int_1^2 \left(q_n \omega + \nu \frac{\partial \omega}{\partial n}\right) ds \text{ on any path.}$$

Applied to ~~line plane~~ straight line 1 to 2: $\int_1^2 \left(q_n \omega + \nu \frac{\partial \omega}{\partial n}\right) ds \approx \cancel{\int_1^2 q_n \frac{\partial q_n}{\partial z} dx}$ ~~This is approx.~~

$\cancel{\int_1^2}$ Let line be y direction, $\therefore q_n = q_x$, $\omega = -\frac{\partial q_x}{\partial y} + \frac{\partial q_y}{\partial x}$.

$$\int_1^2 q_n \omega \, ds = \int_1^2 \left(q_x \frac{\partial q_x}{\partial y} dy - q_x \frac{\partial q_y}{\partial x}\right) dy = \frac{1}{2} q_x^2 \Big|_1^2 - \int_1^2 q_x \frac{\partial q_y}{\partial x} dy.$$

Now if lines are nearly parallel, q_y is small, and $\frac{\partial q_y}{\partial x}$ is also small, (x $\perp$ to line $1 \to 2$.) so the second term is nearly 0. Likewise $\nu \int \frac{\partial \omega}{\partial n} ds \sim \nu \int \frac{\partial \omega}{\partial x} ds = \nu \cdot$ Nearly zero.

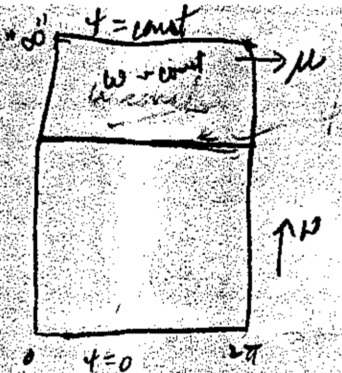
$\therefore \Pi_2 - \Pi_1 = \frac{1}{2} q_2^2 - \frac{1}{2} q_1^2$. $\quad \therefore \underline{p_2 = p_1}$ Valid even with ν, if flow is nearly parallel. Error term is $\int_1^2 q_x \frac{\partial q_y}{\partial x} dy$

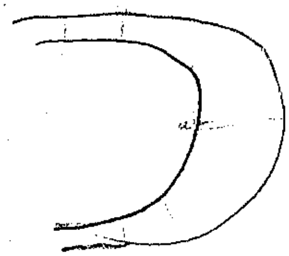
<u>Next</u> If 1 & 2 are also connected by a single stream line,

$$\Pi_2 - \Pi_1 = \int_{1\,\text{streamline}}^2 \nu \frac{\partial \omega}{\partial n} ds \qquad \text{But } \nu \to 0. \therefore \Pi_2 = \Pi_1$$

$\therefore$ For two points such as 1, 2 because they are connected by a single streamline of finite length, we must have $u_2^2 = u_1^2$.

<u>Hence</u> Two points on a streamline which cut a plane far down, have the same square velocity. But there velocity is obviously oppositely directed. Hence they are equal and opposite $u_2 = -u_1$. (This means that the "next" streamline is the same width from ~~one~~ the other, above & below.) $\therefore$ The change in velocity with height is the same at the two stations





Boundry layer eqn:

$$\psi_{\nu\nu}\,\bar{u}^2 = \omega$$

$$\nu\,\omega_{\nu\nu} = \psi_\nu\,\omega_\mu - \psi_\mu\,\omega_\nu$$

ξ,η = Rectangl

$$\psi_\xi = \psi_\mu\bar{u} + \psi_\nu\bar{v}$$
$$\psi_\eta = \psi_\mu\bar{v} - \psi_\nu\bar{u}$$

$$J^2\psi_\mu = \bar{u}\,\psi_\xi + \bar{v}\,\psi_\eta$$
$$J^2\psi_\nu = \bar{v}\,\psi_\xi - \bar{u}\,\psi_\eta$$

Change μ,ν to x,y for convenience. Call $\bar{u}^2 = J^2$ $J(x)$ only

$$J^2\psi_{yy} = \omega$$

$$\nu\,\omega_{yy} = u\omega_x + v\omega_y = \nu J^2\psi_{yyyy} = \psi_x\,J^2\psi_{yyy} - \psi_y\,(J^2\psi_{yy})_x$$

$$= (\psi_x J^2\psi_{yy})_y - \psi_{xy}J^2\psi_{yy} - \psi_y J^2\psi_{xyy} - 2JJ_x\,\psi_y\psi_{yy}$$

$$= (\psi_x J^2\psi_{yy})_y - (\psi_{xy}J^2\psi_y)_y - \frac{J}{?}(J_x\psi_y^2)_y$$

$$\therefore\ \nu J\,\psi_{yyy} = J(\psi_x\psi_{yy} - \psi_{xy}\psi_y) - J_x\psi_y^2$$

$$= J(\psi_x\psi_{yy}) - (J\psi_y)_x\,\psi_y$$

$$= \frac{\partial}{\partial y}(J\psi_x\psi_y) - \frac{\partial}{\partial x}(J\psi_y^2)$$

$\therefore$ av. on x: $\nu\,\overline{J\psi_{yy}}\Big|_y^\infty = \frac{\partial}{\partial x}\overline{J\psi_x\psi_y}\Big|_y^\infty$ Eg $\nu\,\overline{J\omega(0)} = \nu\,\overline{J\omega(\infty)}$

old

$$\nu\psi_{yyy} = \psi_x\psi_{yy} - \psi_{xy}\psi_y$$

Mult by ψ + av.

$$\nu\,\overline{\psi\psi_{yyy}} = \overline{\psi\psi_x\psi_{yy}} - \overline{\psi\psi_{xy}\psi_y}$$

$$= \frac{\partial}{\partial y}\overline{(\psi\psi_x\psi_y)} - \overline{\psi_y^2\psi_x} - \overline{2\psi\psi_{xy}\psi_y}$$

$$\int J\psi_y^2\,dx = \int J\psi_\nu^2\,d\mu$$

$$= \int\frac{1}{J^3}(\bar{v}\psi_\xi - \bar{u}\psi_\eta)^2\frac{d\mu}{d\xi}\,d\xi$$

along streamline.

When old & new coincide, $\bar{v}=0$

$\frac{d\mu}{d\xi} = \bar{u}$, $J = \bar{u}$, $\psi_\xi = v$, $\psi_\eta = u$

$\therefore \int u^2\,d\xi = $ const.

Mult by ψ + av: $\nu\,\overline{J\psi\psi_{yyy}} = \frac{\partial}{\partial y}(\overline{\psi J\psi_x\psi_y}) - \cancel{\overline{J\psi_x\psi_y^2}} - \cancel{\overline{\psi(J\psi_y^2)_x}}$

$$\therefore\ \nu\frac{\partial}{\partial y}\left(\overline{J\psi\psi_{yy}} - \tfrac{1}{2}\overline{J\psi_y^2}\right) = \frac{\partial}{\partial y}(\overline{\psi J\psi_x\psi_y})$$

$$\therefore\ \overline{J\psi\psi_{yy}}\Big|_0^\infty = \tfrac{1}{2}\overline{J\psi_y^2}\Big|_0^\infty \qquad \therefore\ \overline{J\psi_\infty\omega_\infty} = \tfrac{1}{2}\overline{J\psi_y^2}\Big|_\infty - \tfrac{1}{2}\overline{J\psi_y^2}\Big|_0$$

In limit, $\psi_\infty \sim \sqrt{\nu}$, $\omega_\infty \sim$ const $\therefore\ \overline{J\psi_y^2}$ = same just inside as just outside boundry layer.

This means $\int q_t^2\,ds$ is same inside & out!

$$\nu J\psi_{yyy} = J\psi_y\psi_{yx} - J\psi_{yy}\psi_x + J_x\psi_y^2 - J\psi_{xy} - 2J_x\psi_y = \frac{\partial}{\partial x}(J\psi_y^2 - 2J\psi_y) - \frac{\partial}{\partial y}(J\psi_y\psi_x - J\psi_x)$$

(1) $$\nu\overline{J\psi_{yyy}} = -\frac{\partial}{\partial y}(\overline{J\psi_y\psi_x} - \overline{J\psi_x})$$

$$\nu\overline{J\psi_{yy}} - \nu\overline{J\psi_{yy}}\Big|_\infty = \overline{J\psi_y\psi_x} - \overline{J\psi_x}$$

Hence the problem is to find

$$\int J\,\psi_\mu\,d\mu\,d\nu = \int J(\bar{u}\psi_\xi + \bar{v}\psi_\eta)\,d\xi\,d\eta = \int J(\bar{q}\times q)\,d\text{area}$$

$$= \int |\bar{q}|(\bar{q}\times q)\,d(\text{area})$$

Try for 2nd order $\quad \omega = J^2\psi_{yy}$

$$\nu\omega_{yy} = \psi_y\omega_x - \psi_x\omega_y$$

$\psi = -y + \psi' + \psi''$ $\qquad \omega' = J^2\psi'_{yy} \qquad \omega'' = J^2\psi''_{yy} \qquad u' = \psi'_y$ etc.

$\nu\omega'_{yy} = -\omega'_x \quad \nu\bar{\omega}'_{yy} = 0 \quad \therefore \nu\bar{\omega}' = 0 \quad \therefore \overline{J^2\psi'_{yy}} = 0 \quad \overline{J^2\psi'_y} = \overline{J^2 u} = $ const.

$$\nu\omega''_{yy} = \omega''_x + \psi'_y\omega'_x - \psi'_x\omega'_y = \omega''_x + \frac{\partial}{\partial x}(\psi_y\omega) - \frac{\partial}{\partial y}(\psi_x\omega)$$

~~$\nu\bar{\omega}''_y = -\overline{\psi'_x\omega'} = \overline{\psi_x J^2\psi_{yy}} = -\frac{\partial}{\partial y}\overline{\psi_x J^2\psi_y} - \overline{2JJ_x\psi_y^2}$~~

$$= (J^2\psi''_{yy})_x + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\left(\frac{J^2}{2}\psi_y^2\right) - \frac{\partial}{\partial y}(\psi_x\omega)$$

$$\nu\omega''_y = (J^2\psi''_y)_x + JJ_x(\psi'_y)^2 - \psi_x J^2\psi_{yy} + J^2\psi_y\psi_{xy}$$

$$\frac{\nu\omega''_y}{J} = (J^2\psi''_y)_x - \frac{\partial}{\partial y}(\psi_x J\psi_y) + \frac{\partial}{\partial x}(J\psi_y^2) + J_x\psi''_y$$

$-\frac{\partial}{\partial x}(J\psi''_y)$

$-\frac{\partial}{\partial y}(J\psi_x)$

$-\psi'_x$

$$\int J^2\bar{\psi}'_\mu\,d\mu = \int(\bar{q}\cdot q)\,d\mu$$
$$= \int(\bar{q}\cdot q)\,\bar{q}_t\,ds = 0$$
$$\text{or } \int\bar{q}_t^2\,q\,ds = 0?$$

$$-\overline{J^2\psi_x\psi_{yy}} = \frac{d}{dy}\overline{(J^2\psi'_x\psi'_y)} + \overline{J^2\psi_{xy}\psi_y} - \overline{2\psi_y^2 JJ_x}$$

$\omega'_y\psi'_y$

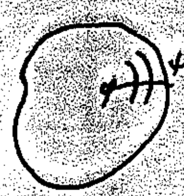
$$\nu\omega'_y = -(J^2\psi'_y)_x + (J^2\psi'_{y0})_x$$

$$-\overline{\psi_x J^2\psi_{yy}} = -\frac{\partial}{\partial y}\overline{J^2\psi_x\psi_y} - \overline{2JJ_x\psi_y^2}$$

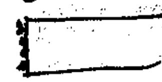
$$W''_{yy} = W''_x + S(x,y)$$ $w'' \to 0$ at ∞. fuel conditions

$$S(x,y) =$$

R. P. FEYNMAN PAPERS



$\nu(\omega_{\psi\psi} + \cancel{\omega_{\varphi\varphi}} = \omega_{\varphi})$ at $\psi = 0$ u is const given, but suppose ω were given



$\bar{\omega}_{\psi\psi} = 0 \quad \bar{\omega}_{\psi} = \text{const} = \omega_0$

Write $\omega(\varphi, 0) = \cancel{e^{ik\varphi}} \sum_n a_n e^{in\varphi}$

$$\omega(\varphi,\psi) = \sum_m a_m e^{im\varphi} e^{-\sqrt{mi}\,\psi}$$

This solves $\nu\nabla^2\omega = (\vec{V}\cdot\nabla)\omega$. But we want to know what q_t is on the surface. $\omega = \nabla\times q = -u_x + v_y$. $\cancel{\nabla^2 u = 2\omega}$

$$\frac{\partial q_t}{\partial n} = \omega + \frac{\partial q_n}{\partial t} + \frac{q_t}{R}.$$

$$u_2 - u_1 = \int_1^2 \omega\, dy = \int_1^2 \omega \frac{\partial y}{\partial \psi} d\psi = \frac{1}{u}\int_1^2 \omega\, d\psi.$$

$d\psi / d\psi/dy$

$$= u^{-1}(\varphi) \cdot \cancel{\omega(\varphi,\psi)} \sum a_n \frac{e^{in\varphi}}{\sqrt{n}}$$

$\therefore \oint \frac{\delta u_2 \bar{u}}{dt} d\varphi = 0$ = criterion. But $u = \frac{\partial\psi}{\partial y} = -\frac{\partial\varphi}{\partial x}$

$$= \oint \delta u_2 \frac{v\, d\varphi}{\frac{\partial\varphi}{\partial x}} = \oint \delta u_2 \hat{u}^2 dx = 0.$$

$$(\bar{U}^2 + \bar{V}^2)\,\varphi(\cancel{u_{\varphi\varphi}} + \cancel{u_{\psi\psi}}) = \omega$$

$\cancel{\oint \frac{\omega}{U^2+V^2}}$

$$\underline{\oint \omega\, d\varphi = 0} \quad \therefore \oint_{\text{streamline}} \omega\, \bar{q}_t\, ds = 0. = \oint \frac{\partial q_t}{\partial n} \bar{q}_t\, ds$$

$$= \oint \frac{\partial q_t}{\partial \psi}\frac{\partial\psi}{\partial n}\bar{q}_t\, ds = \int \frac{\partial q_t}{\partial\psi}\bar{q}_t^2\, ds$$

$J = U^2 + V^2$

$$\therefore \int q_t \bar{q}_t^2\, ds = \text{CONST}$$

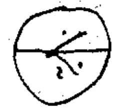
$$\nu(\omega'_{\psi\psi} + \cancel{\omega_{\varphi\varphi}}) = \omega'_{\varphi} + \frac{1}{J}(V_0\cdot\nabla)\omega = \omega'_{\varphi} + S(\psi,\varphi).$$

$$\nu\omega'_{\psi\psi} = \overline{S(\psi,\varphi)} \qquad \nu\bar{\omega}'_{\psi} = \int_0^{\psi} \overline{S(\psi)}\, d\psi \qquad \nu\bar{\omega}' = \int_0^{\psi}\int_0^{\psi'} \bar{S}(\psi'')\, d\psi''\, d\psi' \;\Big|\; \nu\bar{u}' = \int_0^{\psi}\int_0^{\psi'}\int_0^{\psi''} \bar{S}(\psi''')\, d\psi'''\, d\psi''\, d\psi'.$$

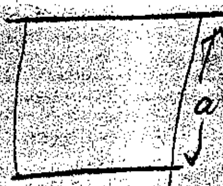
$$S = \frac{1}{J}(V\cdot\nabla)\omega \qquad \int_0^{\psi} S\, d\psi\, d\varphi = \int_0^{\psi} SJ\, dx\, dy = \int \nabla\cdot(V\omega) = \int (V_n\omega)_{\lambda s} = \int q_n \omega\, ds = \oint \frac{q_n\omega}{\sigma} d\varphi$$

Velocity along central axis

$$V_x = 2\int \omega(x',y')\left[\frac{-y'}{r_1^2} - \frac{a^2 y'}{(x'^2+y'^2)r_1^2}\right]dx'dy' \qquad r_2^2 = r_1^2.$$



$$= 2\int \omega(x',y')\left[1 - \frac{a^2}{x'^2+y'^2}\right]\cdot\frac{dx'dy'}{x'^2+y'^2-2(xx'\cancel{+yy'})+x^2}$$



$$\nu \frac{\partial^2 \omega}{\partial y^2} = u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} \quad \omega = \frac{\partial u}{\partial y} \quad u \text{ specified}$$

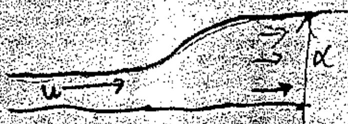
$u_x = v_y \qquad \omega = \nabla^2\psi \approx \psi_{yy}$

change y by α

x by β

ω by $\gamma = \beta/\alpha^3 \qquad \frac{\gamma}{\alpha^2} = \frac{\gamma\alpha^2}{\beta} ~~+ \frac{\gamma\alpha^2}{\beta}\frac{\gamma}{\alpha}~~ \qquad \gamma = \frac{\beta}{\alpha^3}$

u by $\gamma\alpha = \beta/\alpha^2$



v by $\frac{\gamma\alpha^2}{\beta} = \frac{1}{\alpha}$ $\qquad$ ~~$\beta/\alpha^2 = \frac{1}{\alpha}\cdot\frac{\beta}{\alpha^3}\alpha$~~

ψ by $\beta/\alpha = 1 \quad \therefore \beta = \alpha. \qquad \alpha \bar{q}_t = 1$

New coord: $x \to \varphi(x)$

~~$y \to \psi(x,y)$~~

~~$\frac{\partial}{\partial x} = \varphi' \frac{\partial}{\partial \varphi} + \frac{\partial\psi}{\partial x}\frac{\partial}{\partial \psi} \qquad \frac{\partial}{\partial y} = \frac{\partial \psi}{\partial y}\frac{\partial}{\partial \psi}$~~

~~$= \varphi_x \frac{\partial}{\partial \varphi} + \psi_x \frac{\partial}{\partial \psi} \qquad \frac{\partial}{\partial y} = \psi_y \frac{\partial}{\partial\psi}$~~

~~$\nu\nabla^2 u = u u_x + v u_y + p_x \qquad p_x = -\psi_y\psi_{yx}$~~

μ is like φ

ν is like ψ

$\nu \nabla^2 \omega = (q \cdot \nabla)\omega \qquad$ ~~$q = \nabla \times \omega$~~

$\psi(x,y) = \psi(\mu,\nu)$

$-\frac{\partial \nu}{\partial y} = \bar{u} = \frac{\partial \mu}{\partial x}$

$\frac{\partial \nu}{\partial x} = \bar{v} = \frac{\partial \mu}{\partial y}$

$\psi_x = \psi_\mu \bar{u} + \psi_\nu \bar{v} = -v$

$\psi_y = \psi_\mu \bar{v} - \bar{u}\psi_\nu = u$

$\nabla^2\mu = \nabla^2\nu = 0$

$J = \bar{u}^2 + \bar{v}^2$

$\nu\nabla^2\omega = u\omega_x + v\omega_y \qquad \omega = \nabla^2\psi = (\psi_{\mu\mu} + \psi_{\nu\nu}) J$

$u\omega_x + v\omega_y = \psi_x\omega_y - \psi_y\omega_x = (\psi_\mu \bar{u} + \psi_\nu \bar{v})(\omega_\mu \bar{v} - \bar{u}\omega_\nu) - (\omega_\mu \bar{u} + \omega_\nu \bar{v})(\psi_\mu \bar{v} - \bar{u}\psi_\nu)$

$= (\psi_\nu \omega_\mu - \psi_\mu \omega_\nu)(\bar{v}^2 + \bar{u}^2)$

$$\therefore \boxed{\begin{aligned} &\nu(\omega_{\mu\mu} + \omega_{\nu\nu}) = (\psi_\nu\omega_\mu - \psi_\mu\omega_\nu) \\ &(\psi_{\mu\mu} + \psi_{\nu\nu})(\bar{u}^2 + \bar{v}^2) = \omega \end{aligned}}$$

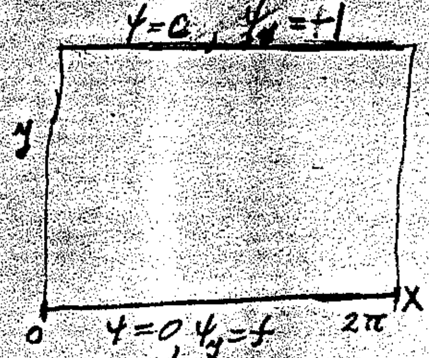
Exact in orthogonal curvilinear coordinates, μ, ν

ψ is old ψ

ω is old ω.

$\bar{u}^2 + \bar{v}^2 = (\nabla\mu)^2 = (\nabla\nu)^2$

$=$ Jacobian



$$\omega = J^2\psi_{yy}$$

$$\nu\omega_{yy} = \psi_y\omega_x - \psi_x\omega_y$$

$$\nu J\psi_{yyy} = \frac{\partial}{\partial x}(J\psi_y^2) - \frac{\partial}{\partial y}(J\psi_y\psi_x) - J_x$$

Write $\psi_{true} = -y + \psi$ $\qquad \omega = J^2\psi_{yy}$

$$\nu\omega_{yy} = \psi_y\omega_x - \psi_x\omega_y - \omega_x$$

$$\nu J\psi_{yyy} = \frac{\partial}{\partial x}(J\psi_y^2 - 2J\psi_y) - \frac{\partial}{\partial y}(J\psi_y\psi_x - J\psi_x)$$

$$-\overline{J\psi_x\psi_y^2} + \overline{\psi\frac{\partial}{\partial y}J\psi_x}$$

$$\nu\overline{J\psi_{yy}} = -\overline{J\psi_y\psi_x} + \nu\overline{J}\,\overline{\psi}_{yy\infty}$$

$$\overline{2\psi(J\psi_y)_x} = -\overline{\nu J\psi\psi_{yyy}} - \overline{J\psi\psi_{yy}\psi_x} - \overline{J\psi\psi_y\psi_{xy}}$$

$$= \frac{\partial}{\partial y}\left[\overline{(\psi\psi_{yy})}\,\nu - \frac{\nu}{2}\overline{\psi_y^2}\right] - \frac{\partial}{\partial y}\left[\overline{J\psi\psi_y\psi_x}\right]$$

$$+\frac{\partial}{\partial y}\left[\overline{J\psi_x\psi}\right] - \overline{J\psi_x\psi_y}$$

$$\therefore\ \nu\overline{J\psi_{yy}} - \nu\overline{J}\,\overline{\psi}_{yy\infty} = -\overline{J\psi_y\psi_x} = \frac{d}{dy}\left[\overline{\nu\psi\psi_{yy}} - \frac{\nu}{2}\overline{\psi_y^2} - \overline{J\psi\psi_y\psi_x} + \overline{J\psi\psi_x}\right]$$

Integrating from 0 to ∞

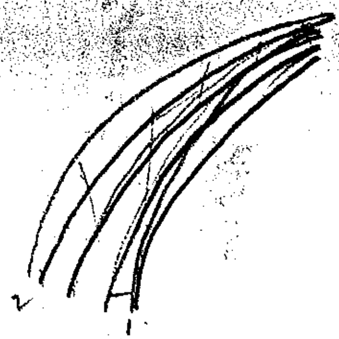
$$\overline{\nu J\psi_y}\Big|_\infty - \nu\overline{J}\,\overline{\psi}_{yy\infty}y_\infty - \overline{\nu J\psi_y}\Big|_0 = \overline{\nu J\psi\psi_{yy}}\Big|_\infty - \frac{\nu}{2}\overline{J\psi_y^2}_\infty - \frac{\nu}{2}\overline{J\psi_y^2}\Big|_0$$

suppose we use limit for ψ_y', ψ_x' $\qquad \nu\omega_{yy} = \omega_x \qquad \omega = J^2\psi_{yy}$

$$\nu\overline{\omega}_y = 0 \qquad \overline{\nu\omega} = 0, \qquad \overline{J^2\psi_{yy}} = 0 \qquad \overline{J^2\psi_y} = \text{const.}$$

$$\int J^2\psi_\nu\,dp = \text{const.}$$

$$\int \frac{\Psi}{p}(\Psi - q)\frac{d\psi}{ds}\,ds = \text{const}$$



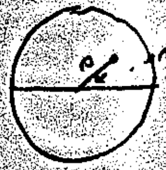
$$\int_1^2 w\,d\mu\,d\nu = 0 = \int_1^2 \omega J^2\,dx\,dy$$

$$\oint \omega J^2 w\,ds = 0 = \oint \frac{\partial q_t}{\partial n}\,\bar{q}_t^2\,w\,ds$$

$$\oint \frac{\partial q_t}{\partial n}\,\bar{q}_t\,ds = 0 = \int \frac{\partial q_t}{\partial \psi}\frac{\partial \psi}{\partial n}\,\bar{q}_t\,ds$$

$$\frac{\partial}{\partial \psi}\int (q_t\,\bar{q}_t^2)\,ds = 0!$$

Velocity Profile In Circle



$$V_x = 2\int \omega(x', y')\left[\frac{y-y'}{r_1^2} - \frac{y - \frac{a^2 y'}{x'^2+y'^2}}{r_2^2}\right] dx'dy'$$

On circle, $r_2^2 = \frac{a^2}{x'^2+y'^2}\, r_1^2$

$$V_{\text{tang at circle}} = 2a\int_0^a\int_0^{2\pi} \omega(\rho,\alpha)\left(1-\frac{\rho^2}{a^2}\right)\frac{\rho\, d\rho\, d\alpha}{a^2 - 2a\rho\cos(\alpha-\beta)+\rho^2}$$

Example: $\omega = +\omega_0$ for α from 0 to π
$-\omega_0$ for α from ~~0 to π~~ $-\pi$ to 0.

$\therefore$ if $\alpha = \beta + \gamma$ then $\omega = +\omega_0$ for $\gamma = -\beta$ to $\pi-\beta$

$\omega = -\omega_0$ for $\gamma = -\beta$ to $-\pi-\beta = -\pi + (-\beta$ to $\pi-\beta)$

$$\int_{-\beta}^{\pi-\beta}\frac{d\gamma}{a^2-2a\rho\cos\gamma+\rho^2} = \frac{2}{a^2-\rho^2}\left[\tan^{-1}\frac{(a^2-\rho^2)\tan\frac{\gamma}{2}}{a^2-\rho^2}\right]_{-\beta}^{\pi-\beta} \quad ((a^2+\rho^2)^2 - 4a^2\rho^2)^{1/2} = a^2-\rho^2.$$

$$= \frac{2}{a^2-\rho^2}\left[\tan^{-1}\left(\frac{a-\rho}{a+\rho}\right)\frac{1}{\tan\frac{1}{2}\beta} + \tan^{-1}\left(\frac{a-\rho}{a+\rho}\right)\tan\tfrac{1}{2}\beta\right].$$

subtract.

$\gamma' = \gamma + \pi,\ \gamma = \gamma' - \pi$

$$\int_{-\beta}^{\pi-\beta}\frac{d\gamma'}{a^2-2a\rho\cos(\gamma'-\pi)+\rho^2} = \frac{2}{a^2-\rho^2}\left[\tan^{-1}\left(\frac{a+\rho}{a-\rho}\,\frac{1}{\tan\frac{1}{2}\beta}\right) + \tan^{-1}\left(\frac{a+\rho}{a-\rho}\tan\tfrac{1}{2}\beta\right)\right]$$

$$= \frac{2}{a^2-\rho^2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{a-\rho}{a+\rho}\right)\tan\tfrac{1}{2}\beta + \tan\frac{a+\rho}{a-\rho}\tan\tfrac{1}{2}\beta\right]$$

$$\therefore V_t = \frac{8a\omega_0}{a^2}\int_0^a \rho\, d\rho\left[\tan^{-1}\left(\frac{a-\rho}{a+\rho}\right)\tan\tfrac{1}{2}\beta - \tan\left(\frac{a+\rho}{a-\rho}\tan\tfrac{1}{2}\beta\right)\right]$$

$$= \frac{8\omega_0}{a}\int_{-a}^{a}\rho\, d\rho\tan^{-1}\left(\frac{a-\rho}{a+\rho}\tan\tfrac{1}{2}\beta\right)$$

$$\int_{-a}^{0}\tan\tfrac{a-\rho}{a+\rho}\, f(\rho)\,\rho\, d\rho = \int_a^0 f(-\rho)\rho\, d\rho = -\int_0^a f(-\rho)\rho\, d\rho.$$

$$= 8\omega_0 a X \qquad X = \int_{-1}^{+1} u\, du\tan^{-1}\left(\frac{1-u}{1+u}\,t\right) \qquad t = \tan\tfrac{1}{2}\beta.$$

Non-Uniform Boundry

at height y boundry curve is $\frac{\Psi}{A(x)}$ $Y = \frac{c}{A(x)}$ $\frac{\Psi}{A(x)}$

$A(x)$ = velocity at boundry in limit $= U$

$V = -y A_x(x)$

$U_x = -V_y \quad \therefore \quad A_x(x) + u_x = + A_x - v_y \qquad \therefore u_x = -v_y$

slope of line, $\frac{V}{U} = -y\frac{A_x}{A}$

$$\nu U_{yy} = U U_x + V U_y + p_x$$

$p + \frac{1}{2}U^2 + \frac{1}{2}V^2 = \text{const.} \quad \therefore p_x = -UU_x$ (because VV_x is order y^2, neglect).

$$\nu u_{yy} = A(x) u_x + A_x u - y A_x u_y + v u_y \overset{+uu_x}{\cancel{+ y A_x y A_{xx}}} \cancel{-v y A_{xx}} \cancel{- y^2 A_x A_{xx}}$$

$\frac{dy}{d\psi} = y\frac{A_x}{A}$

$\therefore y = c/A.$

Orders: $y \sim \sqrt{\nu}$, $A_x x \sim 1$, $v \sim \sqrt{\nu}$, $A \sim 1$, $u \sim 1$

$U_y = +V_x = yA_{xx}$

$\therefore$ to order ν:

$$\nu u_{yy} = A(x) u_x + A_x u - y A_x u_y + v u_y + uu_x$$

$$u_x = -v_y.$$

φ const, ψ const

New variables $U = \frac{\partial\varphi}{\partial x} = +\frac{\partial\psi}{\partial y} \quad \therefore \psi = -yA(x) \qquad A(x) = B_x \quad \psi = -yB_x$

$V = \frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x} \qquad \varphi = B(x) + \frac{y^2}{2}A(x) \qquad \varphi = B + \frac{y^2}{2}B_x$

New variables $\psi = +yB_x$; $\varphi = B(x)$ $\quad \frac{\partial f(\varphi\psi)}{\partial x} = f_\varphi A + y f_\psi A_x = f_x$

$\frac{\partial f}{\partial y} = +f_\psi A(x)$; $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \cancel{f_\psi} - \frac{\partial}{\partial y}[f_\psi(\varphi\psi)A(x)]$

$= -\frac{\partial}{\partial\psi}[\quad](-A) - \cancel{\frac{\partial}{\partial\varphi}[\quad]0}$

$= A^2 f_{\psi\psi}.$

$\cancel{\nu A^2 f_{\psi\psi}}$ $\nu A^2 u_{\psi\psi} = A^2 u_\varphi \cancel{+ yAA_x u_\psi} + A_x u \cancel{+ uu_\varphi A + uyu_\psi A_x}$

$\cancel{-yA_x A u_\psi} + v A(x) u_\psi + uu_\varphi A + yuu_\psi A_x.$

CONTINUITY

$u_\varphi A + y A_x u_\psi = -v_\psi A$

$\frac{\partial}{\partial\varphi}(A(x)u) = Au_\varphi + \frac{A_x}{A}u$

A is funct of φ only!

$$\nu A^2 u_{\psi\psi} = A^2 u_\varphi + A_x u + A(v u_\psi + uu_\varphi) + yA_x uu_\psi.$$

$$= A^2 u_\varphi + A_x u + A(v u_\psi - u v_\psi) \cancel{+ yA_x uu_\psi}$$

$$= A\frac{\partial}{\partial\varphi}(Au) + A(v u_\psi - v_\psi u) \cancel{+ yA_x uu_\psi}$$

$\therefore \cancel{\nu\lambda_{\psi\psi} = \lambda_\varphi + \frac{v}{A}\lambda_\psi - \left(\frac{v}{A}\right)_\psi \lambda} \quad \lambda = Au. \quad \left(\frac{v}{A}\right)_\varphi =$

$$\nu A^2 u_{\psi\psi} = A\frac{\partial}{\partial\varphi}(Au) + A(v u_\psi - v_\psi u)$$

$$I = \int_{-1}^{+1} u\,du \tan^{-1}\left(\frac{1-u}{1+u}t\right) \quad ; \quad t = \tan\tfrac{1}{2}\beta$$

$$d\left(\frac{1-u}{1+u}\right) = d\left(-1+\frac{2}{1+u}\right) = \frac{-2}{(1+u)^2}$$

$$= \frac{u^2}{2}\tan^{-1}\left(\frac{1-u}{1+u}t\right)\Big|_{-1}^{+1} + \int_{-1}^{+1}\frac{u^2}{2}\frac{du\,t}{(1+u)^2+(1-u)^2t^2}$$

$$= -\frac{\pi}{4} + \int_{-1}^{+1}\frac{u^2\,du}{u^2+1+2u\left(\frac{1-t^2}{1+t^2}\right)}\;\frac{t}{(1+t^2)}$$

$$\frac{1-t^2}{1+t^2} = \frac{\cos^2\frac{1}{2}\beta-\sin^2\frac{1}{2}\beta}{\cos^2+\sin^2} = \cos\beta = c.$$

$$\frac{t}{1+t^2} = \frac{\sin\frac{1}{2}\beta}{\cos\frac{1}{2}\beta}\cos^2\tfrac{1}{2}\beta = \cancel{\tfrac{1}{2}(1+\cos\beta)} = \tfrac{1}{2}\sin\beta. = \frac{s}{2}$$

$$= -\frac{\pi}{4} + \frac{s}{2}\int_{-1}^{+1}\frac{u^2\,du}{1+2uc+u^2}$$

$a=1,\ c=1,\ b=2c \qquad \sqrt{4ac-b^2} = \sqrt{4-4c^2} = 2\sin\beta$

$$= -\frac{\pi}{4} + \frac{s}{2}\left[u - \frac{2c}{2}\ln(1+2uc+u^2) + \frac{4c^2-2}{2}\cdot\frac{2}{2s}\tan^{-1}\left(\frac{2u+2c}{2s}\right)\right]_{-1}^{+1}$$

$$= -\frac{\pi}{4} + \frac{s}{2}\left[2 - c\ln\left(\frac{1+c}{1-c}\right) + \frac{2c^2-1}{s}\cdot\left[\tan^{-1}\left(\frac{1+c}{s}\right)+\tan^{-1}\left(\frac{1-c}{s}\right)\right]\right]$$

$\tan^{-1}\frac{u+c}{s}$ $\quad$ β note $\frac{\pi}{2}$

$u=+1 \quad \frac{\pi}{2}-\frac{1}{2}\beta$

$u=0 \quad \frac{\pi}{2}-\beta$

$u=-1 \quad -\frac{1}{2}\beta$

$$= -\frac{\pi}{4} + \cancel{(1+c)-(1+c)}c + s - sc\ln(\operatorname{ctn}\tfrac{1}{2}\beta) + \frac{(c^2-s^2)}{2}\left[\frac{\pi}{2}\cancel{-\beta}\right]$$

$$= -\frac{\pi}{2}\sin^2\beta + \sin\beta - \sin\beta\cos\beta\ln(\operatorname{ctn}\tfrac{1}{2}\beta)$$

Near $\frac{\pi}{2}$: $\beta = \frac{\pi}{2}-\epsilon$

$\operatorname{ctn}\frac{\beta}{2} = \tan\left(\frac{\pi}{4}+\frac{\epsilon}{2}\right) =$

~~$= +\sin\beta\{\ 1\ \frac{\pi}{2}\sin\beta$~~

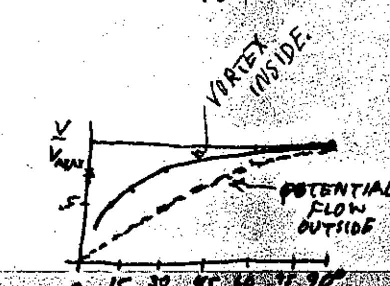
$$V_t = -8\omega_0 a\sin\beta\left[\frac{\pi}{2}\sin\beta + \cos\beta\ln(\operatorname{ctn}\tfrac{1}{2}\beta) - 1\right]$$

$\ln = 2.30\log.$

$1.57\sin\beta + 4.60\cos\beta\log\operatorname{ctn}\frac{1}{2}\beta - 1$

76 $\quad \frac{114}{78}$ $\quad \frac{160}{54}$ $\quad \frac{239}{3}$ 166

β	$1.57\sin\beta$	$4.60\cos\beta\log\operatorname{ctn}\frac{1}{2}\beta$	[ ]	V_t/V_{max}
10°	.272	4.78	~~4.05~~ 1.66	.51
20°	.54	3.21	~~2.75~~ 1.14	.69
30°	.78	2.27	~~2.05~~ .92	.8[illegible]
40°	1.01	1.52	~~1.53~~ .77	.87
50°				
60°	1.35	.55	~~0.90~~ .62	~~0.84~~ .94
70°				
80°				
90°	1.57	0	~~0.57~~ 0.57	~~0.57~~ 1.00



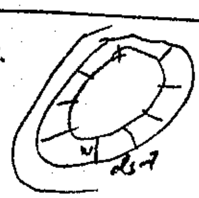
An Integral Theorem in ~~Viscous~~ Steady Viscous Flow ~~in 2 dimensions~~

$\nu\nabla^2 q = (q\cdot\nabla)q - \nabla p. \qquad \nabla\cdot q = 0 \qquad q = \nabla\times\omega$

$\therefore \int q\cdot(\nabla^2 q)\,dVol = 0 = -\int q\cdot(\nabla\times\omega)\,dVol. \quad \therefore \int \omega^2 dVol = \int q\, N\cdot(q\times\omega)\,dSurf.$

cons. of dissipation.

Application to two dimensional flow.



Put $q_x = u = \frac{\partial\psi}{\partial y} = \psi_y$

$q_y = v = -\frac{\partial\psi}{\partial x} = -\psi_x$

$\therefore \omega = \nabla^2\psi$

$\int q\cdot(\nabla\times\omega)\,dVol = 0.$ Apply to thin region between two streamlines which are ~~differential~~ adjacent as shown. Let ds = length element. w = separation (is function of s).

$\therefore \oint q\cdot(\nabla\times\omega)\,ds\cdot w = 0$

But q is tangential, and ~~qw~~ qw = const by continuity.

$\therefore \oint (\nabla\times\omega)_{tang}\,ds = 0 \qquad$ But $(\nabla\times\omega)_{tang} = \frac{\partial\omega}{\partial n}$ = Derivative respect to normal.

$\therefore$ THEO. $\oint \frac{\partial\omega}{\partial N}\,ds = 0$ The mean normal derivative of vorticity taken around a streamline is zero.

Suppose ν is very small, and there is a region in which ~~no potential~~ the flow is essentially that of Euler ($\nu = 0$). Then ω is constant on a stream-line, ~~it is therefore~~ Therefore it is a function of ψ only.

So $\frac{\partial\omega}{\partial n} = \frac{\partial\omega}{\partial\psi}\cdot\frac{\partial\psi}{\partial n} = \frac{\partial\omega}{\partial\psi}\,q_{tang}$ But $\frac{\partial\omega}{\partial\psi}$ is constant over the streamline, so

$\int \frac{\partial\omega}{\partial\psi}\,q_t\,ds = 0 = \frac{\partial\omega}{\partial\psi}\int q_t\,ds \quad \therefore$ Either $\frac{\partial\omega}{\partial\psi} = 0$, or $\int q_t\,ds = 0$. But $\int q_t\,ds$ is the circulation. It may be zero for one streamline, but not for two in succession, unless $\omega = 0$. If $\omega = 0$ then $\frac{\partial\omega}{\partial\psi} = 0$. $\therefore \frac{\partial\omega}{\partial\psi} = 0$. $\therefore \omega$ is the same on all stream lines.

THEO. $\therefore$ (Is this in Batchelor) In the free flow region, in the limit $\nu\to 0$, regions whose streamlines can be shrunk to a common point, are regions of constant ω.

$$\iint v u \, dy\, dx = \iint \Big(\overbrace{-\nu U_y + UV + 2\int_{y'}^{\infty} V U_y \, dy}^{-UV + 2\int U_x U}\Big)\, u \, dy\, dx \qquad UU_x + VU_y = (UV)_y + 2UU_x - \cancel{V_y U}$$

$$= \iint (-\nu U_y U - u^2 V)\, dx\, dy + 2\iint \int_{y'}^{\infty} U_x U(x,y')\, dy' \cdot V(x,y)\, dx\, dy.$$

$$= \frac{\nu}{2}\overline{U^2}\Big|_0 - \iint u^2 V\, dx\, dy + \iiint_{y}^{\infty} U(x,y')\, dy \int_0^{y'} U_y(x,y)\, dx\, dy = \frac{\nu}{2}\,\overline{U^2}\Big|_0 \quad \underline{\underline{\text{ANS.}}}$$

$$\nu U_{yy} = U U_x + V U_y -$$

Flow Past a Variable Boundry

Velocity of stream = 1.
Velocity in x-direct = $1+u$
" " y " = v

$$\nu u_{yy} = u u_x + v u_y + u_x$$

$$u_x = -v_y$$

Velocity in x direct at $y=0$ = $f(x)$

~~f(x) = Periodic. Desire u, v periodic everywhere~~

$f(x)$ = Periodic. period 2π. Problem: To avoid permanent flux into wall some condition on $f(x)$ is required.

Find condition on $f(x)$ so that mean drag on wall is zero:

$$\boxed{\int_0^{2\pi} u_y\,dx = 0.}$$

assume entire solution is periodic.

call $\bar{u}$, $\bar{v}$ functions of y = mean of u; v respect to x. Desire $\bar{u}_y$ at wall: $\nu \bar{u}_y\big|_{y=0} = S$.

$$\nu \bar{u}_{yy} = \overline{v u_y} \qquad \therefore \nu \bar{u}_y\big|_{y=0} = S = \int_0^\infty \overline{v u_y}\,dy = \cancel{vu}\Big|_0^\infty + \int_0^\infty \overline{u_x u}\,dy = 0.$$

$$\nu \bar{u}_y = \int_y^\infty \overline{v u_y}\,dy = (\overline{vu})_y \qquad \therefore \nu \bar{f} = \int_0^\infty (\overline{vu})\,dy.$$

$$u = u^0 + u' + \dots \qquad f = f^0 + f'$$
$$v = v^0 + v' + \dots$$

$$\nu u^0_{yy} = u^0_x \qquad \nu u_{yy} = \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(vu) + u_x$$

$$\nu u'_{yy} - u'_x = u^0 u^0_x + v^0 u^0_y. \qquad \therefore \nu \bar{u}'_{yy} = \overline{v^0 u^0_y}\ ; \ \nu \bar{u}'_y = \overline{v^0 u^0} \qquad \nu \bar{f}' = -\int_0^\infty \overline{vu}\,dy.$$

$$-v^0_y = u^0_x \qquad \therefore v^0 = \int_y^\infty u^0_x(x,y')\,dy' \qquad \therefore \nu \bar{f}' = -\int_0^\infty dy \frac{1}{2\pi}\int_0^{2\pi} u^0(x,y)\, u^0_x(x,y')\,dy'\,dx$$

$$= -\nu u^0_y$$

$$= -\int_0^\infty dy \int_0^\infty \nu \overline{u^0_y u^0}\,dy = -\frac{\nu}{2}(\overline{u^0})^2\Big|_{y=0}$$

$$\therefore \bar{f}' = -\frac{1}{2}\overline{(f^0)^2} \qquad \therefore \overline{f^2} = \overline{(f^0)^2} + 2\overline{f^0 f'} + \cancel{\overline{f'^2}} = 0 \text{ to order } \epsilon^2!$$

$$\nu u^0_{yy} = u^0_x$$

$$\nu u'_{yy} - u'_x = u^0 u^0_x + v^0 u^0_y$$

$$\nu u''_{yy} - u''_x = u^0 u'_x + u' u^0_x + v^0 u'_y + v' u^0_y$$

In ψ:

Exact Equ $\nu\nabla^2(\nabla^2\psi) = \psi_x\nabla^2\psi_y - \psi_y\nabla^2\psi_x - \nabla^2\psi_x \leftarrow$ True $\psi = y+\psi$

$u = -\psi_y$
$v = \psi_x$
$\omega = \nabla^2\psi$
~~Bound~~

Boundry cond. $\psi \to$ const at ∞
$\psi \to -yf(x)$ as $y\to\infty$
(i.e. $\psi = 0$ at $y=0$
$\psi_y = -f(x)$ at $y=0$).

Orders $y \sim \sqrt{\nu}$, $\psi \sim \sqrt{\nu}$. for boundry layer.

$\therefore \nu\psi_{yyyy} = \psi_x\psi_{yyy} - \psi_y\psi_{xyy} - \psi_{xyy}$

which has an integral (on y):

$$\boxed{\nu\psi_{yyy} = \psi_x\psi_{yy} - \psi_y\psi_{xy} - \psi_{xy}}$$

Boundry cond: $\psi \to a(x)$ at $y\to\infty$.
$\psi \sim -yf(x)$ as $y\to 0$.

BOUNDRY LAYER THEORY.

$\nu\overline{\psi_{yy}} = \overline{\psi_x\psi_y}$ ← (for exact equ too).

$\overline{\psi_x\psi_y} = -\overline{\psi\psi_{xy}} = +\nu\overline{\psi\psi_{yyy}} - \overline{\psi\psi_x\psi_{yy}} + \overline{\psi\psi_y\psi_{xy}}$

$= \nu\frac{\partial}{\partial y}(\overline{\psi\psi_{yy}}) - \nu\overline{\psi_y\psi_{yy}} + \frac{1}{2}\overline{\psi^2\psi_{xyy}} + \frac{1}{2}\overline{(\psi^2)_y\psi_{xy}}$

$\nu\overline{\psi_{yy}} = \overline{\psi_x\psi_y} = \frac{\partial}{\partial y}\left[\nu\overline{\psi\psi_{yy}} - \frac{\nu}{2}\overline{\psi_y^2} + \frac{1}{2}(\overline{\psi^2\psi_{xy}})\right]$

Integrate (from ∞, where $\psi_y = 0$),

$\nu\overline{\psi_y} = \nu\overline{\psi\psi_{yy}} - \frac{\nu}{2}\overline{\psi_y^2} + \frac{1}{2}\overline{\psi^2\psi_{xy}}$.

$\therefore \nu\left(\overline{\psi_y} + \frac{1}{2}\overline{\psi_y^2}\right) = \nu\overline{\psi\psi_{yy}} + \frac{1}{2}\overline{\psi^2\psi_{xy}}$. ANY y! (1)

For $y=0$, get $\overline{(u+\frac{1}{2}u^2)}_{\text{at } y=0} = 0$.

also find

$-\overline{\psi^2\psi_{xy}} = \frac{\partial}{\partial y}\left[\nu\overline{\psi^2\psi_{yy}} - \nu\overline{\psi\psi_y^2} - \frac{1}{3}\overline{\psi^3\psi_{xy}}\right] + \nu\overline{\psi_y^3}$ (1)

If it does any good.

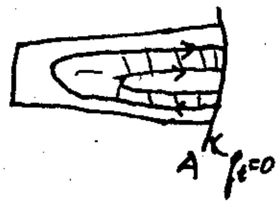
Integrate (1) from 0 to ∞:

$\cancel{\nu\overline{\psi_\infty}} + \frac{3}{2}\nu\int_0^\infty\overline{u^2}\,dy = +\frac{\nu}{2}\overline{u_0^3}$
$\nu\overline{\psi_\infty}$

$$\therefore \boxed{\overline{u_0^3} = \int_0^\infty\overline{(2u+3u^2)}\,dy}$$

SO WHAT?

Incomplete Streamlines TWO DIMENSIONS



$$\nu \int q\cdot\nabla^2 q \, dVol = \int (q\cdot\nabla)\Pi \, dVol \qquad \Pi = p + \tfrac{1}{2}q^2$$

~~Taken~~ $$= \int (q\cdot N)\Pi \, dSurf.$$

Take between two streamlines: $\qquad -\nu\nabla\times\omega = -q\times\omega + \nabla\Pi.$

$$\nu\int q\cdot(\nabla\times\omega)ds\cdot W = q_n W \Pi\Big|_{in}^{out}$$

$$\nu \int q W \, ds (\nabla\times\omega)_t = q_n W \Pi_{in}^{out} \qquad \therefore \frac{\partial \Pi}{\partial s}\Big|_{\text{along streamline}} = \frac{\partial\omega}{\partial n}$$

Theo. $\boxed{\nu\text{. The rate of change of } \omega \text{ Normal to line} = \text{Rate of change of } \Pi \text{ along line.}}$

What is rate of change of ω along line? $\frac{\partial\omega}{\partial s} = \int (q\cdot N)\omega \, dsurf = \int (q\cdot\nabla)\omega \, dVol.$

But $\nu\nabla^2\omega = (q\cdot\nabla)\omega. \quad \therefore \nu\int\nabla^2\omega \, dVol = \int (q\cdot\nabla)\omega \, dVol$

~~Integrate on volume between stream lines:~~

$$\nu\int \nabla\omega\cdot N \, dsurf = \int (N\cdot q)\omega \, dSurf \quad \text{ON any SURFACE.}$$

but on ~~wall between~~ surface of stream lines:

$$\therefore \nu\int_{\text{OVER LINE.}} \frac{\partial\omega}{\partial n} ds = \int_{\text{over A}} q_n \omega \, dA + \nu\int_{\text{over A.}} \frac{\partial\omega}{\partial n} da$$

If we take two successive lines

$$\nu\,\Delta\left(\int \frac{\partial\omega}{\partial n} ds\right) = (q\cdot W)\cdot \Delta\omega\Big|_{at A.}$$

TWO DIMENSIONS WITH OR WITHOUT ν:

$$\nabla^2\Pi = \omega^2 - q\cdot(\nabla\times\omega)$$

Note $\nu\nabla^2 q = (q\cdot\nabla)q + \nabla p \quad = \nabla\cdot(q\times\omega)$

But $\nabla\cdot q = 0$

$\therefore$ We have an identity: $\qquad u_x = -v_y$

$$\nabla^2 p = -\nabla\cdot((q\cdot\nabla)q)$$

$$= -(uu_x + vu_y)_x - (uv_x + vv_y)_y$$

$$= -u_x^2 - v_x u_y - u_y v_x - v_y^2$$

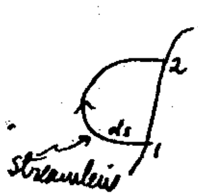
$$\cancel{-uu_{xx} - vu_{yx} - uv_{xy} - vv_{yy}}$$

$$\therefore u_x^2 + 2u_y v_x + v_y^2 = -\nabla^2 p$$

$$+\nabla^2\Pi = +\omega^2 + u\nabla^2 u + v\nabla^2 v$$

With, or without ν!

$\therefore$



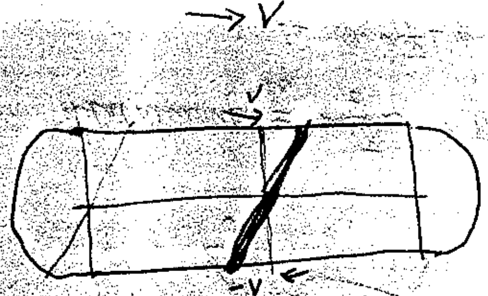
$$\nu\int_1^2 \frac{\partial\omega}{\partial n} ds = \Pi_2 - \Pi_1 = \int_1^2 \left(q_n\omega + \nu\frac{\partial\omega}{\partial n}\right) da$$

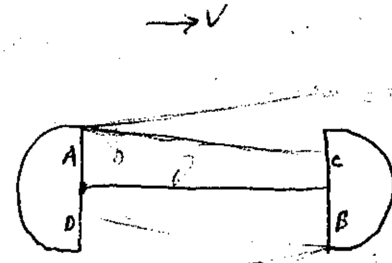
(on streamline) (on line "a")

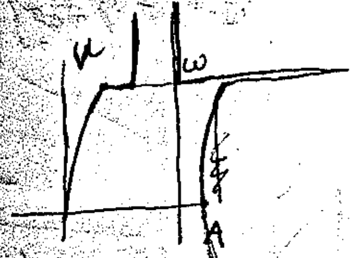
In fact, for any two points and any line

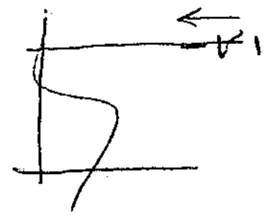
$$\Pi_2 - \Pi_1 = \int_1^2 \left(q_n\omega + \nu\frac{\partial\omega}{\partial n}\right) ds$$

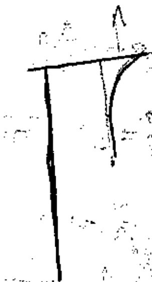
since $\nabla\Pi = -\nu\nabla\times\omega + q\times\omega$

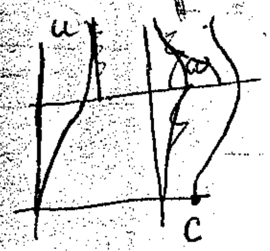












$$\frac{\partial u}{\partial x} = u_{yy} \qquad \frac{\partial w}{\partial x} = w_{yy}$$

Problem at $x=0$ $u = f(y)$ $y = 0$ to L

$u = 1$ $y > 1$

at $x = L$ $u = f(u)$ $y = 0$ to 1

~~$u = ?$ $y > 1$~~ as u beyond

at $y=0$ $u = 0$ all x.

Pulse of heat $\int u\,dy = u_0 - f(1)$

Heat entering $= \frac{1}{2}(u_0 - f(1))$ = Heat leaving.

Case, ∞ semiplane.

$$w(x,y) = \int_{-\infty}^{\infty} \frac{e^{-\frac{(y-y')^2}{2L}}}{\sqrt{2\pi L}}\, w(0,y')\,dy'$$

For $y > 0$

$$f(y) = \int_0^{\infty} \frac{e^{-\frac{(y-y')^2}{2L}}}{\sqrt{2\pi L}}\, f(y')\,dy' + a\,\frac{e^{-y^2/2L}}{\sqrt{2\pi L}}$$

$$f(y) = \int_0^{\infty} K(y-y')\left[f(y') + a\,\delta(y')\right]dy'$$

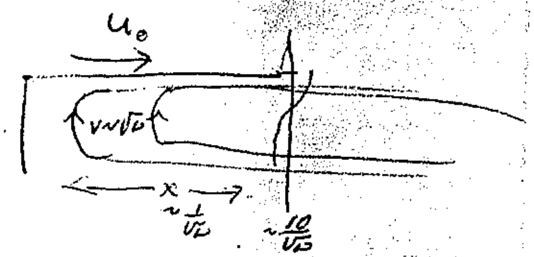
$f = g + \frac{a}{v} e$ $\quad g = \int g + \cancel{\frac{A}{\sqrt{2\pi}} e^{-\frac{y^2}{2L}} \int_{-\infty}^{0} e\, dy'}$

$$\int f^2(y)\,dy - \iint f(y)\, e\, f(y')\,dy' + a \int e f \quad \text{is Min,}$$

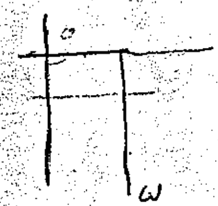
Inner region Equations $x \sim \sqrt{\nu}$, $V \sim \sqrt{\nu}$ $y \sim 1$ ~~(...)~~

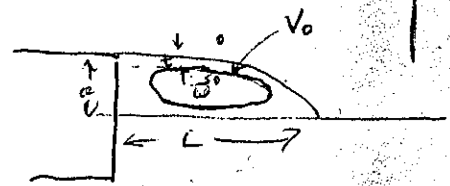
$$\cancel{\nu\,\psi_{yy}}\quad \nu \nabla^2 \omega = u\omega_x + V\omega_y \qquad u_x = V_y$$

$$\omega = u_y - V_x$$



$$\nu\,(u_{yyy} - u_{yxx} - V_{xyy} + \ldots) = \cancel{u u_y}$$
$$= u u_{yx} - u V_{xx} - V u_{yy} + V V_{xy}$$





To bring streamline down, down vel $\sim \frac{a}{L} V$ due to vortex

due to vortex $V_{down} \sim \frac{\bar\omega \cdot aL}{L} = \bar\omega a$

$V_y = -u_x$ $\quad V = \int u_x dy$? $\quad \bar\omega = \frac{V}{L}$? $\therefore L \sim a$!

Heat in/sec $= V(V - V_M)$

Mean vorticity $= \bar\omega$

Flow of $\bar\omega$ out $= \nu \frac{\bar\omega}{t} L$ $\quad t =$ boundary thickness.
/sec

$V_M = \bar\omega a \qquad Vt\bar\omega = \nu \frac{\bar\omega}{t} L$
$= V(V - \bar\omega a)$

$$Vt\bar\omega = \nu \frac{\bar\omega}{t} L = V(V - \bar\omega a)$$

$$t = \sqrt{\frac{\nu L}{V}} \quad \therefore \sqrt{\frac{\nu L}{V}}\,\bar\omega = (V - \bar\omega a)$$

$$\bar\omega = \frac{V}{a + \sqrt{\frac{\nu L}{V}}}$$

Case 1 $\quad \sqrt{\frac{\nu L}{V}} \sim a \quad \therefore L \sim \frac{1}{\nu}$

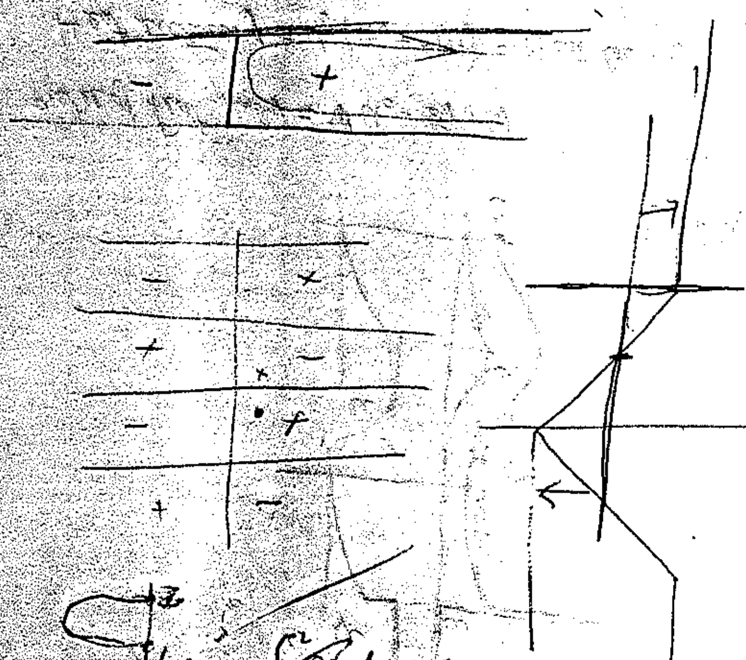
$\bar\omega \approx \frac{V}{a}$ as previous

Case 2 $\quad \bar\omega = \frac{V}{a} - \delta \frac{V}{a}$

~~$\sqrt{\frac{\nu L}{V}} \ll a$~~

$$V\delta a = \sqrt{\frac{\nu L}{V}}\,\frac{V}{a}$$

$$\delta = \frac{1}{a}\sqrt{\frac{\nu L}{V}} = \frac{t}{a}$$



$$\int_1^2 \nu \frac{\partial\omega}{\partial n} ds = \Pi_2 - \Pi_1$$

$\int \ldots p, T, V_1$

Heat removed/sec $= \nu \frac{\bar\omega}{t} L$ $\therefore$ Temp of efflux, if of width w

is $\nu \frac{\bar\omega}{t} L / wV$ $\therefore \delta u_{center} \approx \frac{\nu \bar\omega L}{tV}$

Drag $= \delta u\, W = \frac{\nu \bar\omega L}{tV} W$ But $t \sim W$

$\therefore$ Drag $= \frac{\nu \bar\omega}{V} L = \frac{\nu V}{a} L$ independent of V!?

$f_2(0) = \frac{1}{\sqrt{\pi}}\int_0^\infty f_1(y')\,dy'\,e^{-y'^2}dy' + A$ Write $a = \sqrt{2\pi}\,A$

$= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty dy' e^{-y'^2}\int_{y'}^\infty e^{-y^2}dy + \frac{A}{\sqrt{\pi}}\int_0^\infty e^{-2y^2}dy + A$ old $A = \frac{2}{\pi}$

$= \frac{1}{2} - \frac{1}{2\pi}\frac{\pi}{4} + \frac{A}{2\sqrt{2}} + A$

old $\frac{1}{2} + A_0 = f_1(0)$

$f_2(0) = \frac{3}{8} + A\left(1 + \frac{1}{2\sqrt{2}}\right)$

$A = .637$

$\frac{1}{2} + A = 1.137$

$A = .637$

$\begin{array}{r} 1.353 \\ .637 \\ \hline 2471 \\ 1059 \\ 212 \\ \hline .225 \end{array}$

$\begin{array}{r} .375 \\ .225 \\ .637 \\ \hline 1.237 \end{array}$

$.375 + (1.353)A = .500 + A$

$.353A = .125$

f_1

$\cancel{\int_0^\infty e f_1 dy}$ $f_2(0) = \int_0^\infty f_1\, e\, dy + A$

$\int_0^\infty f_1 e\,dy = .600$ $\int_0^\infty f_0 e\,dy = .500$ $C\left(\int ef\right)_{\text{true}} = \frac{A}{\sqrt{2\pi}}$

$f_1(0) = .$

Problem: to solve $f = \int_0^\infty \frac{e^{-(y-y')^2/2}}{\sqrt{2\pi}} f(y')\,dy' + \frac{a}{\sqrt{2\pi}}e^{-y^2/2}$ ~~put f = 1 - g~~

$\cancel{(1-g) = \int_{-y}^\infty (1-g)e^{-x^2}}$

Observe f minimizes $C = \dfrac{\int_0^\infty f^2 dy - \iint f f e}{\left(\int ef\right)^2}.$

$f(0) = \int ef + \frac{a}{\sqrt{2\pi}}$

$= \left(1 + \frac{C}{\sqrt{\pi}}\right)\int ef$

Desire pulse rise $a = u_0 - u_{\text{sun}} = u_0 - \int_0^\infty (f(y) - 1)\,dy$

$\frac{f(0)}{\int ef}\Big|_{\text{true}} = 1 + \frac{C}{\sqrt{\pi}}$

$y' = y + x$

$\int_0^R f\,dy = -\int_{-y}^{R-y}\int_0^\infty e^{-\frac{x^2}{2}}\frac{dx}{\sqrt{2\pi}} f(y')\,dy'$

$\left(f_0 / \int e f_0\right) = 2$

$f_1(0) / \int e f_1 = .600$

$ae = f - \int e''f' = f - \int Kf$

$f = \int Kf + ae$ choose a so to minimize $f - \int Kf$ ~~$ae(f-Kf)$~~

$f = \int K \int Kf + \int Kae + ae$

$f = \gamma\varphi$ φ given, a given, what is best γ?

$\cancel{x} \; \varphi_1 = \int \gamma\varphi + e$

Minimize $\int(\varphi - \varphi_1)^2 dy \;\therefore\; \int(\varphi - \varphi_1 - e)^2 = \cancel{\int\varphi^2 + \int\varphi}$

Heat pulse $= u_0 - \int (f - f_0)\,dy = a$

Suppose you have a φ which minimizes $\dfrac{\int\varphi^2 - \int\varphi K\varphi}{(\int e\varphi)^2} = C$ C figured

Then φ satisfies $\varphi = \int K\varphi + \left(C\int e\varphi\right)e$

$f = \gamma\varphi$ $f = \int Kf + \left(C\int ef\right)e$

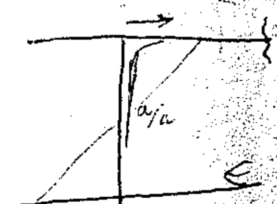
$\therefore a = C\int ef = C\gamma\int e\varphi$

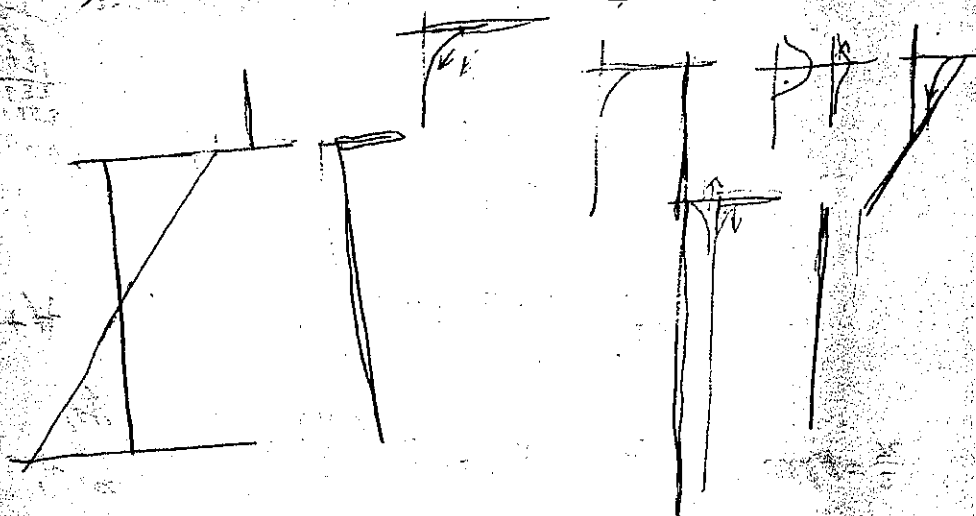
$f_0 = \gamma\varphi_0$

$u_0 - \int(\gamma\varphi - \gamma\varphi_0) = C\gamma\int e\varphi$

$u_0 = \gamma\left[C\int e\varphi - \int(\varphi - \varphi_0)\right]$

$$\frac{u_0}{f_0} = \frac{C\int e\varphi - \int(\varphi - \varphi_0)}{\varphi_0}$$





$$\frac{\int_0^\infty f^2 dy - \int_0^\infty\int_0^\infty \frac{e^{-\frac{(y-y')^2}{2}}}{\sqrt{2\pi}} f(y) f(y') dy dy'}{\left(\int e f\right)^2}$$ find Min. $\underline{\text{Min} = C}$

$$c \int e f dy = a.$$

ans $f - \int K(y,y') f(y') - C e \left(\int e f\right) = 0$

Trial 1. $f = 1$ $\quad \int_0^R \int_0^R \frac{e^{-\frac{(y-y')^2}{2}}}{\sqrt{2\pi}} dy\, dy' = \int_0^R \frac{dy}{\sqrt{2\pi}} \left(\int_{-R+y}^{y} e^{-\frac{x^2}{2}} dx \right)$ $\quad y' = \cancel{x+y+x}$

$$= \frac{1}{\sqrt{2\pi}} \left| y \int_{-R+y}^{y} e^{-\frac{x^2}{2}} dy \right|_0^R - \frac{1}{\sqrt{2\pi}} \int_0^R \left[y\, e^{-\frac{y^2}{2}} dy - \underset{+(R-y)-R}{y}\, e^{-\frac{(R-y)^2}{2}} dy \right]$$

$$= \frac{1}{\sqrt{2\pi}} R \int_0^R e^{-\frac{x^2}{2}} dx - \frac{1}{\sqrt{2\pi}} \left[1 - e^{-R^2/2} + (1 - e^{-R^2/2}) - R \int_0^R e^{-y^2/2} dy \right]$$

$$= R - \frac{2}{\sqrt{2\pi}}$$

$$\int_0^R e f = \frac{1}{2} \quad \therefore \quad C = \frac{2/\sqrt{2\pi}}{1/4} = 8/\sqrt{2\pi} \qquad a = \frac{4}{\sqrt{2\pi}}.$$

$$f_1(y) \approx \int_0^\infty e^{-\frac{(y-y')^2}{2}} \frac{dy'}{\sqrt{2\pi}} + \frac{4}{2\pi} e^{-y^2/2} = \int_{-\infty}^{y} e^{-\frac{\eta^2}{2}} \frac{d\eta}{\sqrt{2\pi}} + \frac{4}{2\pi} e^{-y^2/2}$$

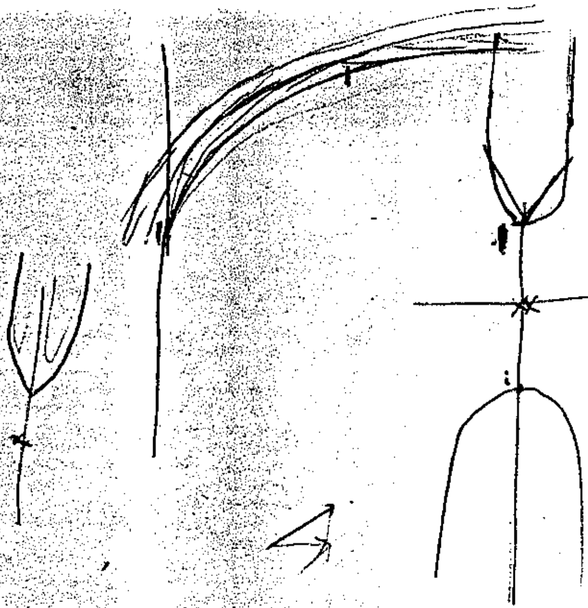
$$= 1 - \int_y^\infty e^{-\frac{\eta^2}{2}} \frac{d\eta}{\sqrt{2\pi}} + \frac{4}{2\pi} e^{-y^2/2}$$

Call $y = \sqrt{2}Y$

$$f_1(Y) = 1 - \frac{1}{2}\left(\frac{2}{\sqrt{\pi}} \int_Y^\infty e^{-\eta^2} d\eta\right) + \overset{a}{\frac{2}{\pi}} e^{-Y^2} \qquad f(0) = 1 - \frac{1}{2} + \frac{2}{\pi}$$

$$f_2(0) = a + \frac{1}{\sqrt{\pi}} \int_0^\infty dY \int_Y^\infty e^{-\eta^2} d\eta + a \int_0^\infty dY e^{-Y^2} + \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\eta^2} d\eta$$

$$= a + \frac{1}{\sqrt{\pi}}$$



$y = \sqrt{x} - \sqrt{a}$

$\delta y = \sqrt{Px}$

$\psi = x^2 + \frac{x}{x^2+y^2} - 1$

$x + \frac{1}{x^2+y^2} = 1 \qquad 2r^2 = xr^2 + 1$

$dx + \frac{2x\,dx + 2y\,dy}{r^4} = 0$ on surf

$v = 2x - 1 + \frac{1}{r^2} + \frac{2x^2}{r^4} = \cancel{2x-2-2+x-2x^2(4-4x+x^2)}$

$u = +\frac{2yx}{r^4}$ on surf.

$\frac{u}{v} = \frac{dx}{dy}\Big|_{surf} = \frac{2y}{r^4(2x-2)-r^2+2x^2} = \frac{2yx}{r^4+2x}$

$(u^2+v^2)_{surf} = x^2 - \frac{8x^3}{r^4} + \frac{4x^2}{r^4}$

$x^2(1+(4-2x)(1-x)^2)$

$v = x - 2x^2(4-2x+x^2)$

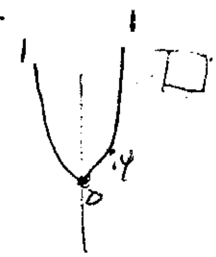
$x=0,\ y=1$

$x=\frac{1}{2},\ y=1.3$

$\frac{1}{2} = \ldots + y^2$

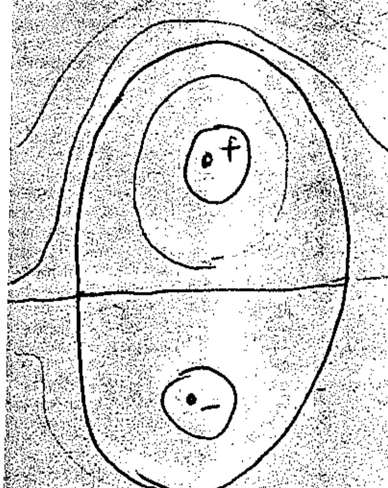
$x + \psi_{Px} = v$

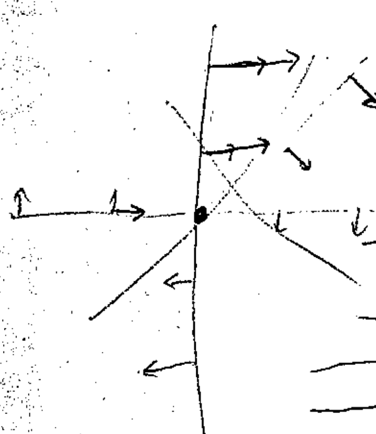
$-\psi_{Py} = u$

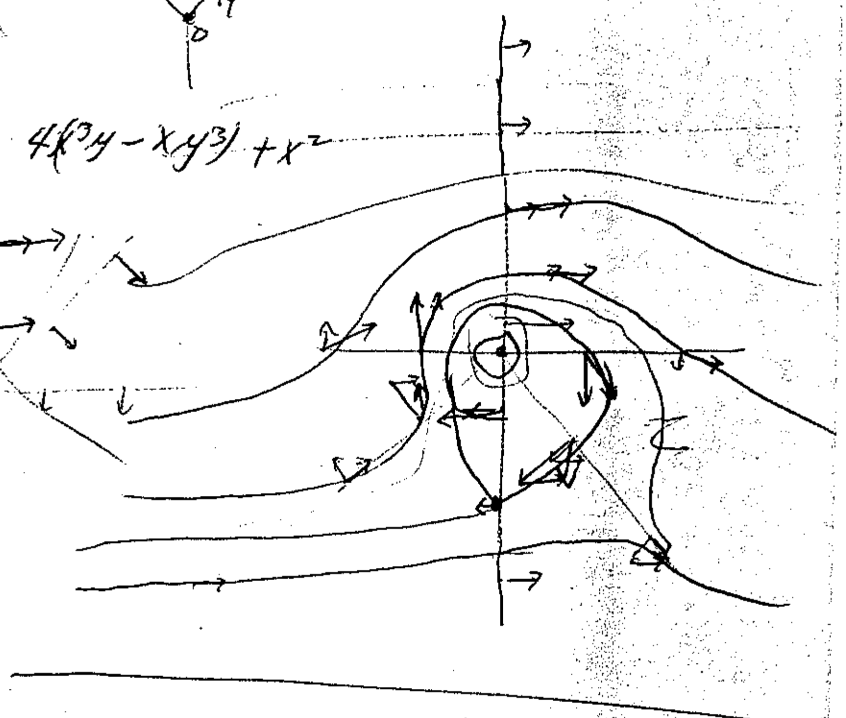


$\cancel{x^3 - 3y^2x + x^2 = 0}$

$x^4 - 6y^2x^2 + y^4 + x^2$ or $4(x^3y - xy^3) + x^2$







$f = \int Kf + ae$

$c = \frac{a}{\int ef}$ Show

$f = 1+g \qquad g = \int Kg + (\int K1 + ae)$

$a \int ef = \int f^2 - \int Kff$

3183
6366
1273
764

$\int ef = \int e \int Kf + \int ae^2$

0

$f_2(0) = \int f_1 e + A$

$\int f_1 e = 0.600$

$c = \frac{8}{\sqrt{2\pi}} = \frac{a}{\int ef_1} = \frac{\sqrt{2\pi}\,A}{\int ef_1} \propto \frac{\sqrt{2\pi}}{.600} A$

$\therefore A = .600 \cdot \frac{4}{\pi} = .764$

$\int ef_1 = f_2(0) - A = \frac{3}{8} + \frac{A}{2\sqrt{2}}$

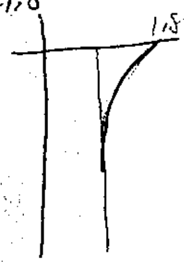
$\frac{8}{2\pi}\left(\frac{3}{8} + \frac{A}{2\sqrt{2}}\right) = A$

$.375 + .353A = .785$

$\begin{array}{r} .785 \\ .375 \\ \hline .410 \end{array}$

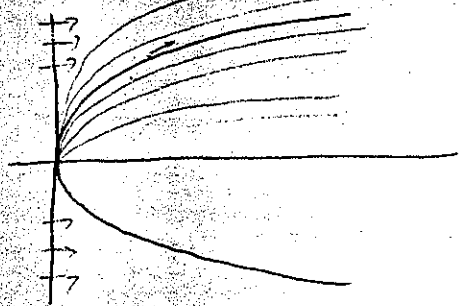
$\underline{A = 1.16}$

$f_2(0) = \frac{3}{8} + 1.16(1.135) = 1.8$



$\delta V \approx$

ψ $\quad y \to \alpha y$, $\psi \to \beta\psi$, $x \to \gamma x$ $\qquad \frac{\beta}{\alpha^2} = \frac{\beta^2}{\gamma\alpha^2}$ or $\beta\alpha = \gamma$

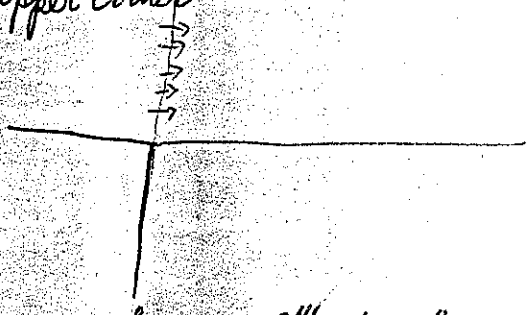


For parabola $y = x^{1/2}$ $\quad \alpha^2 = \gamma$, $\beta = \alpha$

$\psi = \cancel{y}\, f\left(\frac{y}{x^{1/2}}\right)$ $\qquad \cancel{u = f + \frac{y}{x^{1/2}} f'}$

$= x^{1/2} g(z) \qquad z = y/x^{1/2} \qquad u = g'(z)$

Upper Corner



Initial $\psi = y$ $\quad y>0$ at $x=0$ $\qquad g'(z) = 1$ as $z \to +\infty$

$= 0$ $\quad y<0$ at $x=0$ $\qquad = 0$ as $z \to -\infty$

$\psi_x = \frac{1}{2x^{1/2}} g - \frac{1}{2}\frac{y}{x} g' \qquad \psi = x^{1/2} g\left(\frac{y}{x^{1/2}}\right)$

$\therefore \frac{1}{x} g''' = -g'\left(\frac{1}{2x} g' - \frac{1}{2}\frac{y}{x^{3/2}} g'' - \frac{1}{2x} g'\right)$

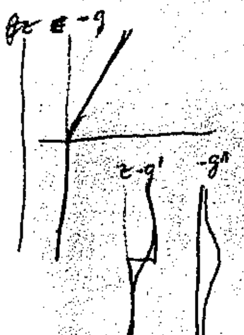
$+ \frac{1}{x^{1/2}} g''\left(\frac{1}{2x^{1/2}} g - \frac{1}{2}\frac{y}{x} g'\right) = \frac{1}{x}\left[+\frac{1}{2} g'' g\right.$

$g''' = \frac{1}{2} g g''$. $\qquad g'' = -A e^{\frac{1}{2}\int^z g\, dz}$

Note: put $g = b f(bz)$
$\therefore$ solve $f(0) = 1$ $\qquad f''' = \frac{1}{2} f f''$.

Trial g: $\quad f = a - bz \quad z>0$
$= a \quad z<0$ $\qquad f''(0) = -1$.

$\int_0^z g\, dz = az - \frac{bz^2}{2} \quad z>0$
$= az \quad z<0$ $\qquad f'' = -e^{\frac{1}{2}\int_0^z f\, dz}$



$f'' = -e^{-\frac{bz^2}{4} + az} \quad z>0$

$= -e^{-a|z|} \quad z<0 \quad = e^{az}$

$f'(z) = -\frac{1}{a} e^{az} \quad z<0$

$= -\frac{1}{a} - \int_0^z e^{-\frac{bz'^2}{4} + az'} dz' \quad z>0 \qquad -f'(z)$ at $\infty = b = \frac{1}{a} + \int_0^\infty e^{-\frac{bz'^2}{4} + az'} dz'$

$\therefore$ One eqn. relates a, b: $\quad b = \frac{1}{a} + \int_0^\infty e^{-\frac{bz^2}{4} + az} dz \qquad f''(z) = -e^{-\frac{bz^2}{4} + az} \quad z>0$

$f''(z) = -e^{+az} \quad z<0$

$g = \frac{1}{\sqrt{b}} f\left(\frac{z}{\sqrt{b}}\right)$ $\qquad$ Get a solution for each b.

Near $z = -\infty$, $g \approx \frac{a}{\sqrt{b}}$; $g' = 0$ $\therefore v = \frac{a}{2\sqrt{b}\, x^{1/2}}$ near $\frac{y}{x^{1/2}} = \infty$

Limit case $a \to 0$

$f = -bz \quad z>0$

$\int_0^z f\,dz = -\frac{bz^2}{2} \quad z>0$

$= 0 \quad z<0$

$f'' = -1 \quad z<0$

$= e^{-\frac{bz^2}{4}} \quad z>0$ impossible.

$$u = g'(z) = -\frac{1}{ab} e^{\frac{a y}{\sqrt{b x}}} \qquad y<0$$

$$= -\frac{1}{ab} - \frac{1}{b^{3/2}} \int_0^{\left(\frac{\sqrt{b} y}{\sqrt{x}}\right)} e^{-\frac{1}{4}\left(\eta - \frac{2a}{\sqrt{b}}\right)^2 + \frac{a^2}{b}} d\eta.$$

~~For small~~ any a $\quad \int_0^\infty e^{-\frac{1}{4}\left(\eta - \frac{2a}{b}\right)^2} d\eta\, e^{\frac{a^2}{b}} + \frac{\sqrt{b}}{a} = b^{3/2}$ Relates a, b.

~~$= \boxed{\sqrt{\pi}\, e^{a^2/b} + \frac{\sqrt{b}}{a} = b^{3/2}}$ relates a, b.~~

For very small a, $b \approx \frac{1}{a}$. $\quad u \approx -1 - \cancel{b}\, a^{3/2} \int_0^{y/\sqrt{ax}} e^{-\frac{1}{4}\eta^2} d\eta \qquad$ ∴ slow spread $y \sim \sqrt{ax}$

$\underset{\text{at} -\infty}{V} \approx \frac{a}{2\sqrt{ax}}$ small, except for $x \sim \frac{1}{a}$

~~(actual V with actual x is $u_0 \frac{\frac{1}{2}\sqrt{a}\sqrt{\nu}}{\sqrt{x}}$)~~

~~Probably $a \sim$ ... ν If we assume~~ Suppose $a \sim 1$ or some no.

∴ ~~vertical velocities go~~ $\quad V \sim \frac{1}{\sqrt{x}} \quad$; or in actual units $\frac{u_0 a}{\sqrt{\frac{a^2 u_0^2}{\nu}}} \frac{1}{\sqrt{x}} = V = u_0 \Big/ \sqrt{\frac{u_0 x}{\nu}}$ (Height of y block)

Hence vertical velocities grow something like $\frac{u_0}{\sqrt{\frac{u_0 x}{\nu}}}$ for $\frac{x}{h} \sim$ small no. $\cdot \left(\frac{h u_0}{\nu}\right)$ $\qquad h =$ Block height.

$\frac{u_0}{\sqrt{\frac{u_0 h}{\nu}}} \cdot \sqrt{\frac{h}{x}}$ if $\sqrt{\frac{h}{x}}\sqrt{\frac{h u_0}{\nu}}$ is large $\qquad$ ~~if $\sqrt{\frac{h}{x}}\sqrt{\frac{h u_0}{\nu}}$ is order 1~~

∴ as long as you are any finite (but very small)(as $\nu \to 0$) fraction of trail length $L = h \frac{h u_0}{\nu}$

V_∞ goes as $\frac{V}{u_0} \sim \frac{1}{\sqrt{x/L}} \cdot \left(\frac{\nu}{u_0 h}\right) \qquad$ ∴ V/u_0 is of order ν, except for x/L too short

∴ ~~$\sqrt{a}$~~. For example if $\frac{x}{L} \sim \sqrt{\frac{\nu}{u_0 h}}$ then $\frac{V}{u_0} \sim \sqrt{\frac{\nu}{u_0 h}} \qquad$ ie for $x \sim$ ~~$\frac{h u_0}{\nu}$~~ $h\sqrt{\frac{h u_0}{\nu}}$

$V \sim u_0 \sqrt{\frac{\nu}{h u_0}}$

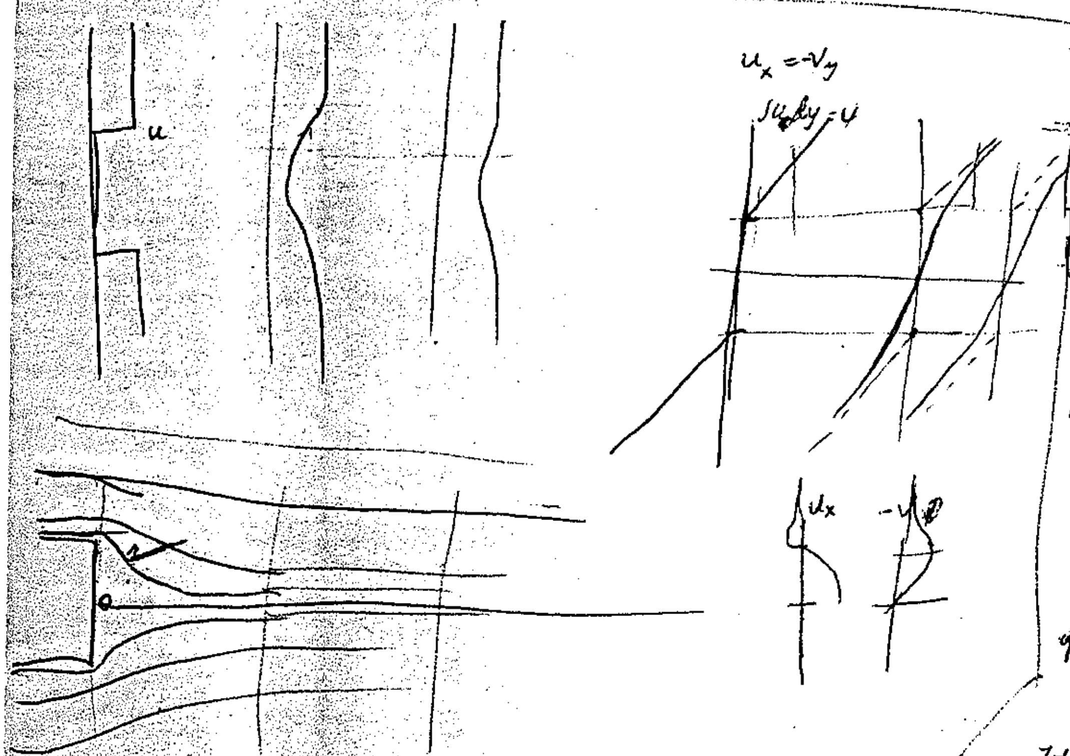
So assumption that $x \sim \frac{1}{\nu}$, $V \sim \nu$ is not self consistent. There is a smaller region which requires study.

Could it be $x \sim \frac{1}{\sqrt{\nu}}$, $V \sim \sqrt{\nu}$?

Pressure in Wake. $p_x + uu_x + vu_y = \nu(u_{xx} + u_{yy})$

~~Kup~~

$$p_x + \psi_y \psi_{xy} - \psi_x \psi_{yy} = -\nu \psi_{yyy}$$



~~Eqn To determine V at incoming surface (if p_x is Negl).~~

Momentum flow across a plane $x = $ const

$= \int_{-\infty}^{\infty} (p + u^2)\,dy + $ Viscous stress $-\int \nu u_x dy = 0$

~~Rate of change with~~

$= -$Drag force $+ \int_{-\infty}^{\infty} (U^2 + p_0)\,dy$.

Q.K. = Const

$\text{Drag} = -\int (p - p_0)\,dy + \int (U^2 - u^2)\,dy$

~~Net flow across plane = 0~~

~~$\int (U - u)\,du = 0$~~

$\int (U-u)(U+u)\,dy = \int (U-u)u\,dy + U\int (U-u)\,dy$

Thus we take $\int_{-a}^{a} u\,dy = 0$

$\therefore \int_{-a}^{a} (U-u)\,dy = +2Ua$

Let $p \approx p_0$. $\text{Drag} = U^2 \cdot 2a$.

Far out $p \approx p_0$ but wake is spread over a width $\sqrt{\nu x}$

But $\int \delta u\,dy = \int (U-u)\,dy = 2aU$ is constant. This looks like a drag $4aU^2$, but there ~~are viscous stress~~ the econ of conservation $\int \delta u\,dy$ is not

There are $-2\nu u_x$ or ~~about νv_y~~

about $\int 2\nu \frac{\delta u}{\delta x}\,dy \approx \frac{2\nu}{x} \cdot 2aU$

exact except for diffusion approx.

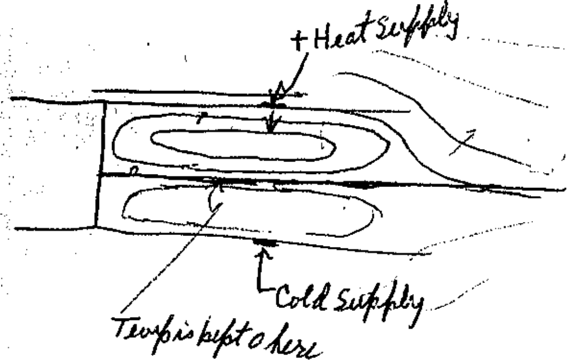
$u_{yy} = -v_x u_y + uv_y + U u_x$ & $\int \delta u\,dy$ is conserved

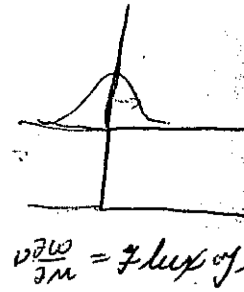
$\therefore \int (U u_x + u_x^2)\,dy$ is conserved just as required. $\text{Drag} = \int U^2 - (U+u)^2$

$= \int (u^2 + 2Uu)\,dy$

But $\int u_x\,dy = v_y\,dy$ $\therefore \int u\,dy = $ const

$(q\cdot\nabla)\omega = \nu\nabla^2\omega$

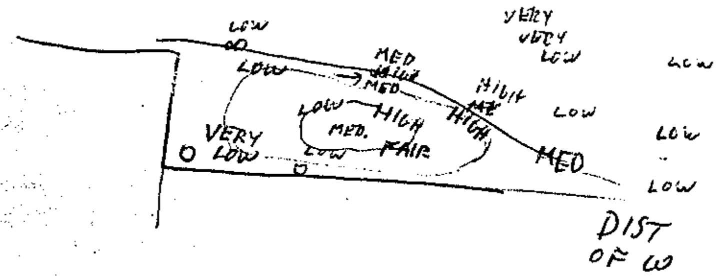


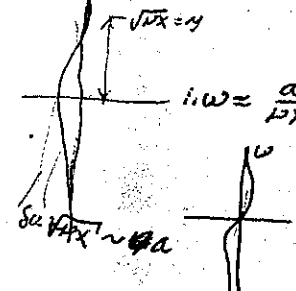


$\frac{ud}{\nu} = R$

$\nu\frac{\partial\omega}{\partial n}$ = Flux of heat/cm/sec

ω = Heat = Temp./cm²

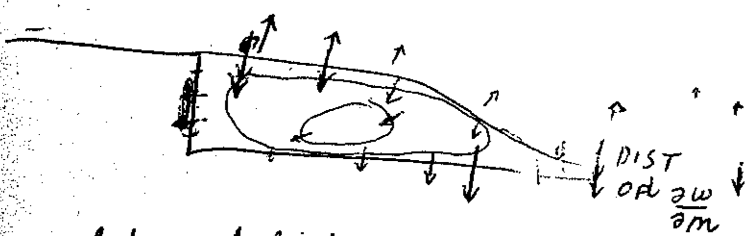




$\delta u\sqrt{\nu x} \sim a$

$\omega = v_x - u_y$

$\int_0^\infty dy\,\omega = \delta u$



$\nu\int_{1\,\text{stream}}^{2} \frac{\partial\omega}{\partial n} = \Pi_2 - \Pi_1$

on any line

$\nu\int_1^2 \frac{\partial\omega}{\partial n}ds + \int_1^2 q_n\omega\, ds = \Pi_2 - \Pi_1$

It is evident that if the picture is qualitatively OK;

1. ω is Not Constant.
2. It is Not constant along a streamline
3. It is transported along x, mainly, but diffuses mainly along y.
4. Just after the block the flow is practically $\omega = 0$. (is this sure?)
5. When the ω gets high, in the back upper tail, the velocity gradients are high — but spread (while at beginning they are high in a very narrow range) The velocities of the stream-lines are larger here too.
6. The central vortex lines are nearly ω = const.
7. We estimate (not too sure) that the central vorticity is about $\frac{V}{L\sqrt{\frac{\nu L}{V}}}$. But it is also $\frac{V}{a}$ $\therefore a = \sqrt{\frac{\nu L}{V}}$
8. So we find the length of the wake is roughly $L = a\left(\frac{aV}{\nu}\right)$ and the penetration thickness is of order a.
9. From all this we conclude that the adequate equation is $u\omega_x + v\omega_y = \nu\omega_{yy}$ with $u(\text{at } x=0) = 0$ for $\frac{y}{a} = a$, $\omega = u_y$; $= 1$ for $y > |a|$

The value we estimate as follows: Heat in $= V$ ~~carried in time t input into~~ stream is put in over about L cm. ~~erase~~ $\therefore VL$ in. This flows thru and out $V\cdot\frac{V}{2}$ cal per sec put in.

Flows out to temp $T = 0$ in time $\frac{L}{V}$. What is temp inside. Thickness from which it flows is $\sqrt{\frac{\nu L}{V}}$ $\therefore$ Mean grad is $T/\sqrt{\frac{\nu L}{V}}$ = Flux/ν cm/s

Flow out/sec $= \frac{\nu L T}{\sqrt{\frac{\nu L}{V}}} = \frac{V^2}{2}$ $\therefore T = \frac{V}{\sqrt{\frac{\nu L}{V}}} = \frac{V}{L\sqrt{\frac{\nu}{LV}}}$

$$u u_{yx} + v u_{yy} = \nu u_{yyy} \quad \cancel{\text{or } \psi_x \psi_{yyy} \quad \psi_y \psi_x} = \frac{\partial}{\partial y}(v u_y - u v_y) \qquad \boxed{u_x = -v_y}$$

$$\therefore \boxed{\nu u_{yy} = v u_y - u v_y} \; + F(x)$$

But as $y \to \infty$ at any station, $\psi \to -y + C$, $u \to 1$, $v \to 0$

This should do $\therefore F(x) = 0$

~~Let us call νx by X, and $\frac{1}{\nu} v$ by V, get~~

$$\cancel{\boxed{u_{yy} = V u_y - u V_y}} \quad \cancel{\boxed{u_x = -V_y}}$$

Initially $V = 0$, $u = \frac{u_0}{R}$ for $y > a$
$(x=0)$ $u = 0$ for $y < a$

$$\boxed{u_y = \int_0^{\infty} 2 \nu u_y^{y} \, V u}$$

Now put $\frac{y}{a} = a\eta$, $u = u_0 u'$, $x = \beta x'$, $V = \frac{u_0 a}{\beta} V'$

$\nu \frac{u_0}{a^2}$ $\frac{u_0 a}{\beta} \frac{u_0}{a}$ $\boxed{\beta = \frac{a^2 u_0}{\nu}}$ $\therefore x$ Now measures in scale $a \cdot \frac{a u_0}{\nu} = a$ Reynolds No $\frac{1}{2}$.

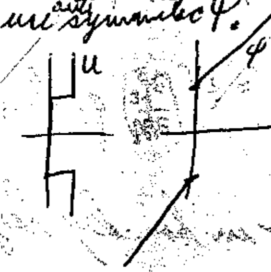
$$\boxed{u_{yy} = V u_y - u V_y} \quad \boxed{u_x = -V_y}$$

Initially $V = 0$
$(x=0)$ $u = 1$ for $y > 1$
$= 0$ for $y < 1$

$\underline{u_y = 0 \text{ for } y = 0}$
, use only $y > 1$
or use cut symmetric ψ.

Check consistency of initial cond:
Not so good. $\therefore V = 0$ Not OK at $y \sim 1$.

Or initial $\psi = -y + 1$ for $y > 1$
$= 0$ for $y < 1$.



How can we get started?

(Example, work near y_0, but call it 0. try $u = \tan^{-1} \frac{y}{\epsilon}$ $\quad \epsilon \to 0$

$u_y = \frac{\epsilon}{\epsilon^2 + y^2}$, $u_{yy} = -\frac{2y\epsilon}{(\epsilon^2 + y^2)^2}$

$$V = u \int_0^y \frac{u_{yy}(y')}{u^2} dy = \frac{u_y}{u} + 2\int_0^y \frac{u_y^2}{u^3} dy)$$

Use ψ $\; -\psi_y = u$. $\; v = \psi_x$ Thus given $\psi_{(x=0)}$ & the equation ψ_x is determined always, & $\therefore V$

$$\boxed{\psi_{yyy} = -\psi_y \psi_{xy} + \psi_x \psi_{yy}}$$

Still, to solve this you must make a sensible start, about ψ_x etc.

$$C_x = u_{yy} - C_y \int_0^y \frac{u_{yy}}{2C} dy$$

$$C = u^2/2$$

$$D_x = \mathcal{E} - D_y \int_0^y \mathcal{E}(y') dy / 2$$

$$D = \ln u$$

$$\mathcal{E} = u_{yy}/u^2 \quad \cancel{\text{or } \mathcal{E}}$$

$$\cancel{u_x = D_{yy} + 2 D_y \int}$$

$$u_x = \mathcal{J}_y + 2\mathcal{J} \int_0^y \mathcal{J}^2/u \, du \quad \text{cannot solve this way.}$$

$$\mathcal{J} = u_y/u$$

$\nu \nabla^2 \omega = (q \cdot \nabla)\omega$.

Mult by ψ & int by parts:

FOR ANY CLOSED PATH $\nu \oint \psi \frac{\partial w}{\partial n} ds - \nu \int (q \times \nabla w)\, d\text{area} = \int \psi\, q_n \omega\, ds$.

$\therefore$ For a streamline, $\int (q \times \nabla \omega)\, d\text{area} = 0. = \oint q_t \omega\, ds - \int \omega^2\, d\text{area}$.

---

also, consider on streamlines ($\bar{q}_t$ is another vector) of $\bar{q}_t$ $\quad q_t = \frac{\partial \psi}{\partial n} \quad \therefore \frac{\partial W}{\partial \psi} = \frac{\partial W}{\partial n}\frac{1}{q_t}$

$$\frac{1}{2}\frac{\partial}{\partial \psi} \oint \bar{q}_t\, q_t^2\, ds = \frac{1}{2}\int \underbrace{\left(\frac{\partial \bar{q}_t}{\partial \psi}\right) q_t^2}\, ds + \int \bar{q}_t\, q_t \frac{\partial q_t}{\partial \psi}\, ds$$

$$= \frac{1}{2}\int \left(\frac{\partial \bar{q}_t}{\partial n}\right) q_t\, ds + \int \bar{q}_t \frac{\partial q_t}{\partial n}\, ds.$$

Now in boundary layer theory the boundary layer has thickness $t \sim \sqrt{\nu}$, so $\Delta\psi \sim \bar{q}_t\, t$. Integrate the above from line inside boundary to outside:

$\Delta \left(\int \bar{q}_t q_t^2\, ds\right) = \int_{in}^{out} \{\text{RHS.}\}\, d\psi$ We wish to show that as $\nu \to 0$ the RHS remains finite, so $\Delta\left(\int \bar{q}_t q_t^2\, ds\right) \to 0$.

$\frac{\partial \bar{q}_t}{\partial n}$ is finite so $\int \frac{\partial \bar{q}_t}{\partial n} q_t\, ds$ is $\therefore$ only question is $\int \bar{q}_t \frac{\partial q_t}{\partial n}\, ds$, which

is sensibly equal to $\oint \bar{q}_t\, \omega\, ds = \int \nabla \times (\bar{q}_t \omega)\, d\text{area} = \int \bar{\omega}\omega\, d\text{area} + \int (\bar{q}_t \times \nabla\omega)\, d\text{area}$

Now $\bar{\omega}$ is constant, so $\int \bar{\omega}\omega\, d\text{area} = \bar{\omega} \oint q_t\, ds$ is finite. $\therefore$ Only last term is dangerous ~~$= \int (\bar{q}_t \cdot \nabla^2 q)\, d\text{area}$~~

But $\oint q_t \times \nabla\omega\, d\text{area} = 0$ So last term $= \int (\bar{q}_t - q_t) \times \nabla\omega\, d\text{area}$.

Integrated form!

$$\nu\,\nabla^2\psi_y = (\nabla^2\psi)\psi_x - \Pi_x$$
$$\nu\,\nabla^2\psi_x = -(\nabla^2\psi)\psi_y + \Pi_y.$$

$$\Pi = p + \frac{1}{2}\frac{L^3U^3}{\nu}\left(\frac{\nu}{L_0^2U}\psi_y^2 + \frac{\nu^2}{L_0^4U^2}\psi_x^2\right)$$

$$\frac{\nu^{5/2}}{L^{9/2}U^{3/2}}\cdot\frac{L^{3/2}U^{3/2}}{\nu^{1/2}}\psi_{yyy} = \frac{L^3U^3}{\nu}\frac{\nu}{L_0^2U}\frac{\nu}{L_0^3U}\psi_{yy}\psi_x - \Pi_x\frac{\nu}{L_0^2U}$$
$$+\frac{L^3U^3}{\nu}\frac{\nu^3}{L_0^6U^3}\psi_{xx}\psi_x$$

$$\frac{\nu^{3}}{L_0^5U^2}\frac{L^{3/2}U^{3/2}}{\nu^{1/2}}\psi_{xyy} = -\frac{\nu^{3/2}}{L_0^{9/2}U^{3/2}}\cdot\frac{L^3U^3}{\nu}\psi_{yy}\psi_y$$
$$+\frac{L^3U^3}{\nu}\cdot\frac{\nu^{1/2}}{L^{7/2}U^{1/2}}\frac{\nu^2}{L_0^4U^2}\psi_{xx}\psi_y + \frac{\nu^{1/2}}{L_0^{7/2}U^{1/2}}\Pi_y$$

Equ 1 $\quad \Pi_x = \psi_{yy}\psi_x$

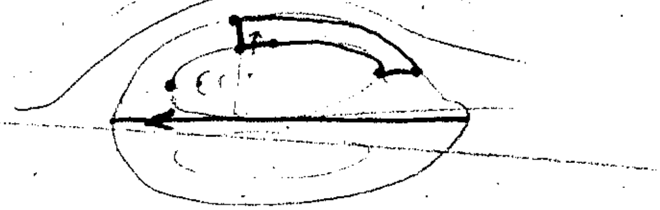
$\Pi_y = \psi_{yy}\psi_y$

$\psi_{yyy} =$

are ①, ② consistent?

$$\cancel{\psi_{yyyy}\psi_x} + \psi_{xy}\psi_{yyy} - \psi_{yy}\psi_{xyy} - \psi_y\psi_{xyyy} + \psi_x^2\psi_{yxx} - \psi_x\psi_y\psi_{xxx} = 0$$

$$-Vu_y - uu_x = f(x) = -(V\cdot\nabla)u$$



Integral of u along stream line

$$\int (V\cdot\nabla)u \, dVol = \int f(x)\, dVol = \int (V\cdot N)u \, dsurf = V_n u_1 \cdot L_1 - V_n u_2 L_2$$

For suff large vertical y,

$\psi \sim y - \delta(x)$

~~$\psi_x = \delta'(x)$~~

$V \to 0$

$\frac{\partial u}{\partial y} \to 0$

$u \to U$

$u_x \to \delta'(x)$?

center stream line,

$V = 0 \quad \therefore -uu_x = f(x)$

$$\frac{d}{dy}\left(\frac{\psi_x}{\psi_y}\right) = -\frac{f(x)}{\psi_y^2}$$

$$\frac{d}{dy}\left(\frac{U}{V}\right) = \frac{f(x)}{u^2}$$

$\Pi = p + \frac{1}{2}(u^2 + v^2)$

$\Pi_x = -\nu\omega_y + \omega V$

$\Pi_y = \nu\omega_x - \omega u$

$\Pi = p + \frac{1}{2}$

Front corner

$\psi = -Uy + C r^{1/2}\cos(\theta/2)$

$\cancel{\sqrt{r + \cos\theta\, r}}$

$C\sqrt{r+y} + A$

$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$

$\frac{1}{4}$

$\cancel{u = U+}$

$x = y^2/L$

$\psi = 0$:

$-2AUy + A^2$

$U^2y^2 = C(r+y)$

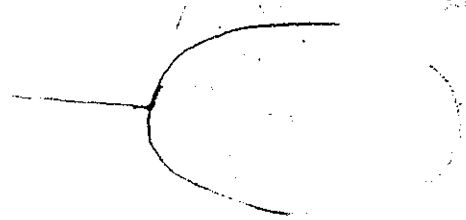
$\cancel{U^2y^2 - Cy = Cr}$

$\cancel{U^4y^4 - 2U^2Cy^3 + C^2y^2 = C^2x^2 + C^2y^2}$

$2AU = -C$

$U^2y^2 + A^2 = Cr$

$U^4y^4 + 2A^2U^2y^2 + A^4 = C^2y^2 + C^2x^2$



$\psi = C\sqrt{r+y}$

$\psi = a$

$r = \frac{a^2}{C} - y$

$x^2 + \cancel{y^2} = \frac{a^4}{C^2} - \frac{2a^2}{C}y + \cancel{y^2}$

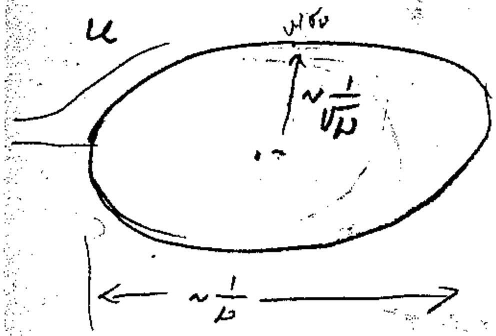
parabola.

(ψ + ν have same dim)

$$\nu\,\nabla^2(\nabla^2\psi) = \psi_x\,\nabla^2\psi_y - \psi_y\,\nabla^2\psi_x$$

$V = \psi_x$

$U = -\psi_y$



$$y' = \sqrt{\frac{\nu}{L_0 u}}\,y/L_0$$

$$x' = \frac{\nu}{L_0 u}\frac{x}{L_0}$$

Force $= \frac{1}{2}u^2 L_0$

$\frac{\nu}{u} = L$

$$y = \frac{L_0^{3/2}u^{1/2}}{\nu^{1/2}}\,y'$$

$$x = \frac{L_0^2 u}{\nu}\,x'$$

$$\frac{y^2}{x} = L_0\,\frac{y'^2}{x'}$$

$u \simeq \psi_y$ $\qquad \psi \sim y u$

$$\psi \sim \sqrt{\frac{\nu}{L_0 u}}\,\frac{u y'}{L_0}$$

$$\psi = \frac{L^{3/2}u^{3/2}}{\nu^{1/2}}\,\psi'$$

∇^2: $\frac{\partial}{\partial x} \rightarrow \frac{\nu}{L_0^2 u}\frac{\partial}{\partial x'}$

$\frac{\partial}{\partial y} \Rightarrow \sqrt{\frac{\nu}{L_0 u}}\cdot\frac{1}{L_0}\frac{\partial}{\partial y'}$

$$\therefore \nabla^2 = \frac{\nu}{L_0^3 u}\frac{\partial^2}{\partial y^2} + \frac{\nu^2}{L_0^4 u^2}\frac{\partial^2}{\partial x^2}$$

$$\frac{\nu^3}{L_0^6 u^2}\psi_{yyyy} + \text{order } \nu^4 = \frac{\nu}{L_0^2 u}\sqrt{\frac{\nu}{L_0 u}}\cdot\frac{1}{L_0}\cdot\frac{\nu}{L_0^3 u}\left(\psi_{yyy}\psi_x - \psi_y\psi_{xyy}\right) + \text{order } \nu\ldots$$

$$\cancel{\frac{\nu^{5/2}u^{1/2}}{L_0^{9/2}u^{+1/2}}\psi_{yyyy}} = \ldots + \frac{\nu^3}{L_0^6 u^3}\sqrt{\frac{\nu}{L_0 u}}\frac{1}{L_0}\left(\psi_{yxx}\psi_x - \psi_y\psi_{xxx}\right)$$

$$\psi_{yyyy} = \frac{\nu^{3/2}u^{1/2}}{L_0^{3/2}}\left(\psi_{yyy}\psi_x - \psi_y\psi_{xyy}\right) + \frac{\nu^{5/2}}{L_0^{9/2}u^{1/2}}\left(\psi_{yxx}\psi_x - \psi_y\psi_{xxx}\right)$$

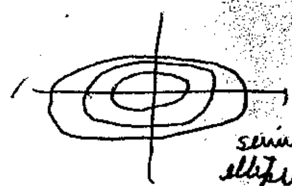
$\therefore$ 2 eqn ① $\psi_{yyy}\psi_x = \psi_y\psi_{xyy}$ $\qquad \therefore \boxed{\psi_x\psi_{yy} - \psi_y\psi_{xy} = f(x)}$

② $\psi_{yyyy} = \psi_{yxx}\psi_x - \psi_y\psi_{xxx} + K$

PAGO SYSTEM FOR FINDING BODIES WHOSE SOLUTION WE KNOW

Case 1 Inside is constant $\omega_0 = 1/2$. ∴ actual Solution must be $\psi_{ACT} = x^2 + \psi_P$ where $\nabla^2\psi = 0$. ∴ $\nabla^2\psi_{ACT} = \omega^2$. Now the condition on the surface must be $\psi = $ const, say $\psi = 0$. ∴ Lines of constant $x^2 + \psi$ are lines of possible boundries. actually ψ_{ACT} may come from sources $\nabla^2\psi_{ACT} = \rho$ provided that $\rho = 0$ inside any chosen streamline.

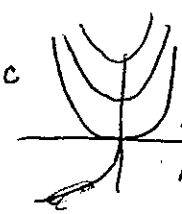
Examples. Take $\psi_p = \beta y^2$ ∴ Curves are $x^2 + \beta y^2 = $ const

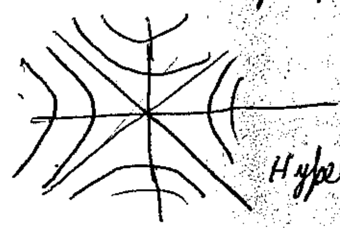


$\beta > 0$

~~$\psi_p = \beta y$~~

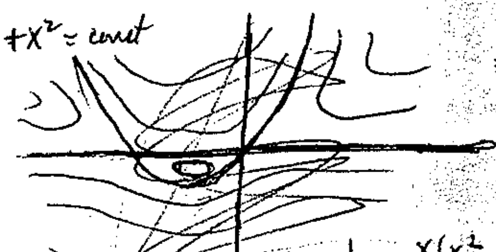
$\psi_p = y$ Curves $x^2 + y = C$



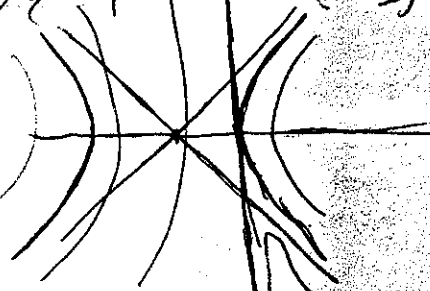


$\psi_p = x^3 - 3y^2x$ Curves $x^3 - 3y^2x + x^2 = $ const

Various blunt nosed objects, pointed objects, and closed ovals, including a parabolic segment.



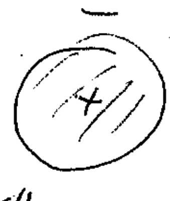
$x(x^2 - 3y^2 + x) = C$



Case 3. Curved boundry. Same idea & technique as straight boundry, but potential problem must be solved for curved boundry. If boundry is closed, some source is required inside & an image outside, so we must be careful with sources.

Example. Circular boundry $\omega = +4$ inside, -4 outside.

$$\psi = r^2 - 1 \quad r<1$$
$$= 1 - r^2 \quad r>1.$$

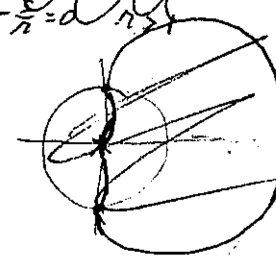


Example dipole source at center $\psi_p = S\frac{x}{r^2} - Sx$ ∴ ~~$r^2 - 1 + \frac{S}{r} = 0 \quad r<1$~~ ~~$r^2 - 1 - \frac{S}{r} = 0 \quad r>1$~~

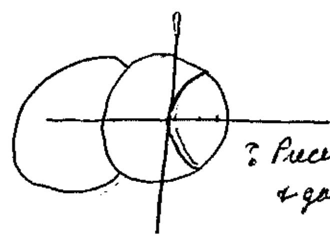
$$r^2 - 1 + \frac{Sx}{r^2} = Sx \quad r<1$$

$$r^2 - 1 - \frac{Sx}{r^2} = Sx \quad r>1$$

$$(r^2-1) + \frac{Sx}{r^2}(r^2-1) = 0 \quad \therefore \; Sx = r^2 \quad r<1$$
$$-Sx = r^2 \quad r>1$$



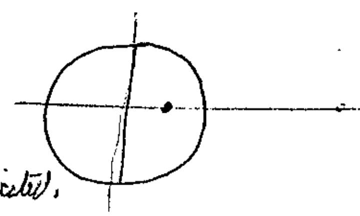
~~Vortex at center $\psi = 0$~~



Source at $x = \frac{1}{2}, y = 0$ $\quad \psi = \frac{x-\frac{1}{2}}{(x-\frac{1}{2})^2+y^2} - \frac{(x-2)}{(x-2)^2+y^2} = \frac{x-\frac{1}{2}}{r^2 - x + \frac{1}{4}} - \frac{x-2}{r^2-4x+4}$

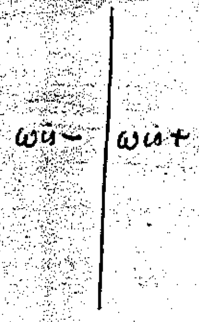
$$\pm(r^2-1) = \frac{x-\frac{1}{2}}{r^2-x+\frac{1}{4}} - \frac{x-2}{r^2-4x+4} \quad r \lessgtr 1$$

$$= \frac{x-\frac{1}{2}}{(r^2-1)-(x-5/4)} - \frac{x-2}{(r^2-1)-4(x-5/4)}$$ too complicated.



Case 2. Boundry of two different ω's. Example Straight boundry. ±ω.

$\omega = 2$

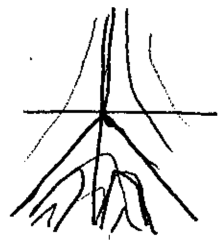


Line is $x=0$. $\nabla^2\psi_V = \pm\omega$. $\psi_V = +x^2$ for $x>0$
$= -x^2$ for $x<0$

Now we require that we find ψ_p subject to $\psi_p = 0$ at wall. (So wall will be stream line too), and the only valid container is the $\psi_p + \psi_V = 0$ line. (The other lines will lie in one region or the other & not intersect the wall — they will be good for case 1 only).

Examples $\psi = \beta xy$ $\therefore x^2 + \beta xy = 0$ $x = -\beta y$

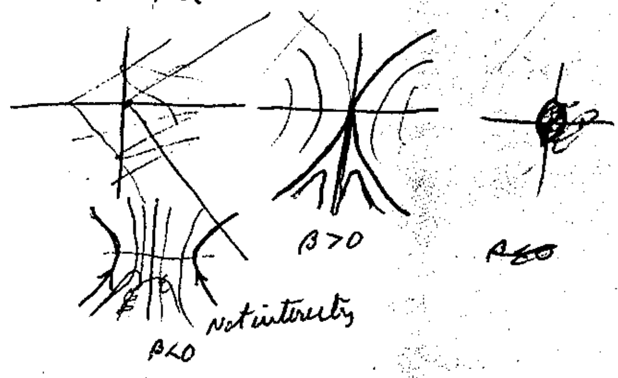
~~$\psi = \beta xy + ax$~~ ~~$x + \beta y + a = 0$~~



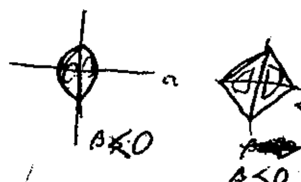
Corner

$\psi = (x^3 - 3y^2x)\beta$ $\therefore x^2 + x^3 - 3y^2x = 0$

or $x + (x^2 - 3y^2)\beta = 0$



$\psi = (x^3 - 3y^2x)\beta + ax$ $x + (x^2 - 3y^2)\beta + a = 0$



rhombus ~~square!~~

$\beta = -1/2$, $a = -2$

Example with a dipole source at origin

$\psi = +\frac{x}{x^2+y^2} \mp ax$ $x + \frac{1}{x^2+y^2} = +a$

$a > 0$ say

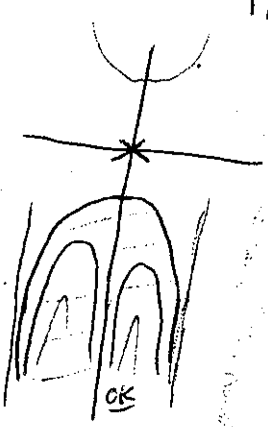
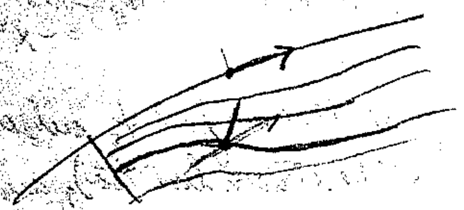
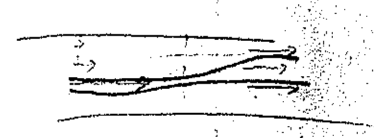


Figure with 11 asymptotes

etc.

$$\int \bar{q}_t \times (\mathrm{def}\, q) \cdot N \, ds$$





$$\int \bar{q}_t \left(\frac{\partial}{\partial n} q_t\right) ds = \int \bar{q}_t \, q_t \frac{\partial q}{\partial \phi} \, ds$$

$$\int \left(\frac{\partial q}{\partial n}\right) q \, ds$$

$$q_t = \nabla \theta$$

$$\int \bar{q}_t \, \omega \, ds$$

$$\nu \nabla \times \omega = \nabla p + (q \cdot \nabla) q$$

$$\nu \int \bar{q}_t \cdot (\nabla \times \omega) = \int (\bar{q}_t \cdot \nabla) p + \int \bar{q}_t \cdot (q \cdot \nabla) q$$

$$\nu \int (q_t \times N) \omega \, ds = \int (\bar{q}_t \cdot N) p + \int (q \cdot \nabla) q_t$$

$$+ \nu \int \bar{\omega} \cdot \omega \, dVol$$